

## Review

- Sums of independent r.v.'s.

(1) Both  $X, Y$  are discrete.

$$\begin{aligned} P_{X+Y}(a) &= \sum_y P_X(a-y) P_Y(y) \\ &= \sum_x P_Y(a-x) P_X(x) \end{aligned}$$

(2) Both  $X$  and  $Y$  are cts.

$$\begin{aligned} f_{X+Y}(a) &= \int_{-\infty}^{\infty} f_X(a-y) f_Y(y) dy \\ &= \int_{-\infty}^{\infty} f_Y(a-x) f_X(x) dx \end{aligned}$$

- Conditional distributions.

1. Suppose both  $X$  and  $Y$  are discrete.

The conditional prob. mass function of  $X$  given  $Y=y$ , is given by

$$P_{X|Y}(x|y) = P\{X=x | Y=y\} = \frac{P(x,y)}{P_Y(y)},$$

provided  $P_Y(y) > 0$ .

2. Suppose that  $X$  and  $Y$  are jointly cts with density  $f(x, y)$ .

Def. The conditional density function of  $X$

given  $Y=y$ , is given by

$$f_{X|Y}(x|y) := \frac{f(x, y)}{f_Y(y)},$$

provided that  $f_Y(y) > 0$ .

Def. For  $A \subset \mathbb{R}$ , the conditional prob. of  $X$  taking values in  $A$  given  $Y=y$  is given by

$$P\{X \in A | Y=y\} = \int_A f_{X|Y}(x|y) dx$$

In particular,

$$\begin{aligned} F_{X|Y}(a|y) &:= P\{X \leq a | Y=y\} \\ &= \int_{-\infty}^a f_{X|Y}(x|y) dx. \end{aligned}$$

Remark: If  $X$  and  $Y$  are independent,

$$\text{then } f_{X|Y}(x|y) = f_X(x).$$

(since in such case  $f(x, y) = f_X(x) f_Y(y)$ )

Remark: One may view

$$P\{X \in A \mid Y = y\}$$

$$= \lim_{\varepsilon \rightarrow 0} P\{X \in A \mid y - \varepsilon < Y < y + \varepsilon\}$$

$$= \lim_{\varepsilon \rightarrow 0} \frac{P\{X \in A, y - \varepsilon < Y < y + \varepsilon\}}{P\{y - \varepsilon < Y < y + \varepsilon\}}$$

Example 1. Suppose the joint density of  $X$  and  $Y$  is given by

$$f(x, y) = \begin{cases} e^{-x/y} e^{-y}/y & \text{if } x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find  $P\{X > 1 \mid Y = y\}$ .

Solution:

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f(x, y) dx \\ &= \int_0^{\infty} e^{-x/y} e^{-y}/y dx \quad (\text{if } y > 0) \\ &= -e^{-x/y} e^{-y} \Big|_{x=0}^{\infty} \\ &= e^{-y} \quad \text{if } y > 0. \end{aligned}$$

Hence for  $y > 0$ ,

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} = e^{-x/y}/y$$

if  $x > 0$ .

Therefore

$$\begin{aligned} P\{X > 1 \mid Y = y\} &= \int_1^{\infty} e^{-x/y} / y \, dx \\ &= -e^{-x/y} \Big|_{x=1}^{\infty} \\ &= e^{-1/y} \quad \text{if } y > 0. \end{aligned}$$



§ 6.7. Joint distributions of functions of r.v.'s.

Setup:  $X_1, X_2$  are joint cts with density  $f_{X_1, X_2}(x_1, x_2)$ . Let  $g_1, g_2: \mathbb{R}^2 \rightarrow \mathbb{R}$ .

Let  $Y_1 = g_1(X_1, X_2)$ ,  $Y_2 = g_2(X_1, X_2)$ .

Find out the joint distribution of  $Y_1, Y_2$

Thm 2. Assumptions: ①  $x_1, x_2$  can be solved  
in terms of  $y_1, y_2$ .

②  $g_1, g_2$  have cts partial derivatives

and

$$J(x_1, x_2) := \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix}$$
$$= \frac{\partial g_1}{\partial x_1} \frac{\partial g_2}{\partial x_2} - \frac{\partial g_1}{\partial x_2} \frac{\partial g_2}{\partial x_1} \neq 0$$

Then  $Y_1, Y_2$  have a joint density

$$f_{Y_1, Y_2}(y_1, y_2) = \underbrace{f_{X_1, X_2}(x_1, x_2)} \cdot |J(x_1, x_2)|^{-1}$$

Example 3. Let  $X_1, X_2$  be jointly cts with  
density  $f(x_1, x_2)$ .

Let  $Y_1 = X_1 + X_2$ ,  $Y_2 = X_1 - X_2$ .

Find the joint density of  $Y_1, Y_2$ .

Solution: Let  $g_1(x_1, x_2) = x_1 + x_2$

$$g_2(x_1, x_2) = x_1 - x_2.$$

• If  $y_1 = x_1 + x_2$   
 $y_2 = x_1 - x_2$ , then  $\begin{cases} x_1 = \frac{y_1 + y_2}{2} \\ x_2 = \frac{y_1 - y_2}{2} \end{cases}$

•  $J(x_1, x_2) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= -2.$$

•  $f_{Y_1, Y_2}(y_1, y_2) = f(x_1, x_2) \cdot |J(x_1, x_2)|^{-1}$   
 $= f(x_1, x_2) / 2$   
 $= f\left(\frac{y_1 + y_2}{2}, \frac{y_1 - y_2}{2}\right) / 2.$

□