Math 3280 20-11-05

Review

- Joint distribution of two r.v.'s
- Independence of two r.v's.

Remark: The above concepts can be generalized to finitely many r.v.'s.

Let $X_1, \cdots, X_n$ be r.v.'s.

① The joint CDF of $X_1, \cdots, X_n$ is

$$F(a_1, \cdots, a_n) = P\{X_1 \le a_1, \cdots, X_n \le a_n\}, \quad a_1, \cdots, a_n \in \mathbb{R}.$$

② Say that $X_1, \cdots, X_n$ are jointly cts if $\exists\, f: \mathbb{R}^n \to [0, \infty)$ such that

$$P\{(X_1, \cdots, X_n) \in C\} = \int \cdots \int_C f(x_1, \cdots, x_n)\, dx_1 \cdots dx_n$$

for all "measurable" sets $C \subset \mathbb{R}^n$.

③ Say $X_1, \cdots, X_n$ are independent if

$$P\{X_1 \in A_1, \cdots, X_n \in A_n\} = P\{X_1 \in A_1\} \cdots P\{X_n \in A_n\}$$

(*) for all $A_1, \cdots, A_n \subset \mathbb{R}$.

Also $(*)$ is equivalent to

$$(**)\quad F(a_1, \cdots, a_n) = F_{X_1}(a_1) \cdots F_{X_n}(a_n), \quad \forall\, a_1, \cdots, a_n \in \mathbb{R}.$$

§ 6.3 Sums of independent r.v.'s

Q: Let X, Y be independent r.v.'s.
How to determine the distribution of $X+Y$?

1. The case when X, Y are both continuous.

Suppose X has a density $f_X(x)$, Y has a density $f_Y(y)$.

Then X, Y have a joint density

$$f(x,y) = f_X(x)\, f_Y(y).$$

For $a \in \mathbb{R}$,

$$F_{X+Y}(a) = P\{X+Y \le a\} = \iint_{\{x+y \le a\}} f_X(x)\, f_Y(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \int_{0}^{a-y} f_X(x)\, f_Y(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} F_X(a-y)\, f_Y(y)\, dy.$$

$$=: F_X * f_Y(a) \qquad ①$$

$$\left(\text{where} \quad g*h(a) := \int_{-\infty}^{\infty} g(a-y)\, h(y)\, dy. \right)$$

In particular

$$f_{X+Y}(a) = \frac{d F_{X+Y}(a)}{da} = \int_{-\infty}^{\infty} \frac{d}{da}\left(F_X(a-y)\right) f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} f_X(a-y)\, f_Y(y)\, dy$$

$$= f_X * f_Y(a). \qquad ②$$

Example 1. If X, Y are independent, and both unif. dist. on $[0,1]$, Calculate the joint density of $X+Y$.

Solution:

$$f_{X+Y}(a) = f_X * f_Y(a)$$

$$= \int_{-\infty}^{\infty} f_X(a-y)\, f_Y(y)\, dy$$

$$= \int_0^1 f_X(a-y)\,dy$$

$$= \int_{a-1}^{a} f_X(z)\,dz .$$

If $0 < a \leq 1$

$$f_{X+Y}(a) = \int_0^a 1\,dz = a$$

If $1 < a \leq 2$

$$f_{X+Y}(a) = \int_{a-1}^{1} 1\,dz = 2-a$$

If $a < 0$ or $a > 2$,

$$f_{X+Y}(a) = 0 .$$

i.e.

$$f_{X+Y}(a) = \begin{cases} a & \text{if } 0 < a \leq 1 \\ 2-a & \text{if } 1 < a \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

Example 2. Let X, Y be independent normal r.v.'s with parameters $(0,1)$ and $(0,\sigma^2)$, respectively. Find the density of $X+Y$.

Solution.

$$f_{X+Y}(a) = f_X * f_Y(a)$$
$$= \int_{-\infty}^{\infty} f_X(a-y)\, f_Y(y)\, dy$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(a-y)^2}{2}} \cdot \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{y^2}{2\sigma^2}}\, dy$$
$$= \frac{1}{2\pi\sigma} \int_{-\infty}^{\infty} e^{-\frac{(a-y)^2}{2} - \frac{y^2}{2\sigma^2}}\, dy$$

Notice that

$$\frac{(a-y)^2}{2} + \frac{y^2}{2\sigma^2} = \frac{\left(\sqrt{\sigma^2+1}\, y - \frac{a\sigma^2}{\sqrt{\sigma^2+1}}\right)^2}{2\sigma^2} + \frac{a^2}{2(\sigma^2+1)}$$

Hence

$$f_{X+Y}(a) = \frac{1}{2\pi\sigma} e^{-\frac{a^2}{2(\sigma^2+1)}} \int_{-\infty}^{\infty} e^{-\frac{\left(\sqrt{\sigma^2+1}\, y - \frac{a\sigma^2}{\sqrt{\sigma^2+1}}\right)^2}{2\sigma^2}} dy$$

$$\left(\text{Letting } z = \frac{\sqrt{\sigma^2+1}\, y - \frac{a\sigma^2}{\sqrt{\sigma^2+1}}}{\sigma}\right)$$

$$= \frac{1}{2\pi\sqrt{\sigma^2+1}} e^{-\frac{a^2}{2(\sigma^2+1)}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz$$

$$= \frac{1}{\sqrt{2\pi(\sigma^2+1)}} e^{-\frac{a^2}{2(\sigma^2+1)}}.$$

Hence $X+Y$ is a normal r.v. with parameters $(0, \sigma^2+1)$.

<u>Remark</u>: In general, if X, Y are ind. normal r.v.'s with parameters (μ_1, σ_1^2), (μ_2, σ_2^2). Then $X+Y$ is a norm r.v. with parameters $(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$.

2. The case that X, Y are both discrete.

In this case

$$P\{X+Y=k\} = \sum_y P\{X=k-y\}\,P\{Y=y\}$$

$$= \sum_y p_X(k-y)\,p_Y(y)$$

$$= \sum_x p_X(x)\,p_Y(k-x).$$

Example 3. Suppose that X, Y are independent Poisson r.v.'s with parameters λ_1, λ_2, resp. Find the distribution of $X+Y$.

Solution: Notice that X, Y both take values in $0, 1, 2, \cdots$, Hence the range of $X+Y$ is the set $\{0, 1, 2, \cdots\}$

For $n = 0, 1, 2, \cdots$

$$P\{X+Y=n\} = \sum_{k=0}^{n} P\{X=k\}\,P\{Y=n-k\}$$

$$= \sum_{k=0}^{n} \cdot e^{-\lambda_1} \cdot \frac{\lambda_1^k}{k!} \; e^{-\lambda_2} \cdot \frac{\lambda_2^{n-k}}{(n-k)!}$$

$$= \frac{e^{-\lambda_1-\lambda_2}}{n!} \cdot \sum_{k=0}^{n} \cdot \binom{n}{k} \lambda_1^k \lambda_2^{n-k}$$

$$= \frac{e^{-\lambda_1-\lambda_2}}{n!} \cdot (\lambda_1+\lambda_2)^n.$$

Hence, $X+Y$ is a Poisson r.v. with parameter $\lambda_1+\lambda_2$.

§ 6.4. Conditional distributions

1. Discrete case.

Def. Let X, Y be two discrete r.v.'s.

The conditional prob. mass function of X given $Y=y$ is

$$p_{X|Y}(x|y) := P\{X=x \mid Y=y\} = \frac{P\{X=x, Y=y\}}{P\{Y=y\}} = \frac{p(x,y)}{p_Y(y)},$$

if $p_Y(y) \neq 0$.

Example 4. If X, Y are independent Poisson r.v's with parameters λ_1, λ_2. Calculate the conditional distribution of X given $X+Y=n$.

Solution:

$$P\{X=k \mid X+Y=n\}$$

$$= \frac{P\{X=k,\ X+Y=n\}}{P\{X+Y=n\}}$$

$$= \frac{P\{X=k\}\ P\{Y=n-k\}}{P\{X+Y=n\}}$$

$$= \frac{e^{-\lambda_1}\frac{\lambda_1^k}{k!} \cdot e^{-\lambda_2} \cdot \frac{\lambda_2^{n-k}}{(n-k)!}}{e^{-\lambda_1-\lambda_2}\frac{(\lambda_1+\lambda_2)^n}{n!}}$$

$$= \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}, \quad \text{for } k=0,1,\cdots,n.$$

Hence the conditional distribution of X

given $X+Y=n$

is the binomial distribution with parameters

n and $\frac{\lambda_1}{\lambda_1+\lambda_2}$.