Math 3280 20-11-02

Review.

- Let X, Y be two r.v.'s. The joint CDF of X, Y is

$$F(a,b) = P\{X \leq a, Y \leq b\}, \quad \forall a, b \in \mathbb{R}$$

- Say X and Y are jointly cts if $\exists f : \mathbb{R}^2 \to [0,\infty)$ such that

$$P\{(X,Y) \in C\} = \iint_C f(x,y)\,dx\,dy,$$

for all "measurable" sets $C \subset \mathbb{R}^2$.

We call f the joint density of X and Y.

- Say X and Y are independent if

$$P\{X \in A, Y \in B\} = P\{X \in A\}\, P\{Y \in B\},$$
$$\forall A, B \subseteq \mathbb{R}.$$

Equivalently

$$F(a,b) = F_X(a)\, F_Y(b), \quad \forall a, b \in \mathbb{R}.$$

Remarks

- Relation between joint CDF and joint density.

$$F(a,b) = P\{X \le a, Y \le b\}$$
$$= \int_{-\infty}^{b} \int_{-\infty}^{a} f(x,y)\, dx\, dy.$$

In the case that f is cts, then

$$\frac{\partial F(a,b)}{\partial b} = \int_{-\infty}^{a} f(x,b)\, dx$$

$$\frac{\partial^2 F(a,b)}{\partial a\, \partial b} = f(a,b).$$

- Equivalent def of independence for r.v.'s.

Lem 1. Suppose X and Y are discrete. Then

X and Y are independent

$$\Longleftrightarrow \quad p(x,y) = p_X(x)\, p_Y(y) \qquad (*)$$

Pf. Clearly X and Y are independent

$$\Leftrightarrow \quad P\{X\in A, Y\in B\} = P\{X\in A\}\cdot P\{Y\in B\}.$$

Letting $A=\{x\}$, $B=\{y\}$ gives

$$p(x,y) = p_X(x)\, p_Y(y).$$

Now suppose (*) holds for all x, y,

Then for given $A, B \subset \mathbb{R}$,

$$\begin{aligned} P\{X\in A, Y\in B\} &= \sum_{x\in A}\sum_{y\in B} p(x,y) \\ &= \sum_{x\in A}\sum_{y\in B} p_X(x)\, p_Y(y) \\ &= \Big(\sum_{x\in A} p_X(x)\Big)\Big(\sum_{y\in B} p_Y(y)\Big) \\ &= P\{X\in A\}\, P\{Y\in B\}. \quad \square \end{aligned}$$

Lem 2. If X and Y are jointly continuous, then X and Y are independent

$$\Leftrightarrow \quad f(x,y) = f_X(x)\, f_Y(y).$$

Pf. X and Y are independent

$$\Leftrightarrow \quad F(a,b) = F_X(a)\, F_Y(b), \quad \forall\, a, b \in \mathbb{R}$$

$$\Rightarrow \quad \frac{\partial^2 F(a,b)}{\partial a\, \partial b} = \frac{d F_X(a)}{d a} \cdot \frac{d F_Y(b)}{d b}$$

i.e. $$f(a,b) = f_X(a)\, f_Y(b). \quad (**).$$

Now if $(**)$ holds, then

$$F(a,b) = \int_{-\infty}^{b} \int_{-\infty}^{a} f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{b} \int_{-\infty}^{a} f_X(x)\, f_Y(y)\, dx\, dy$$

$$= \left(\int_{-\infty}^{b} f_Y(y)\, dy \right) \left(\int_{-\infty}^{a} f_X(x)\, dx \right)$$

$$= F_Y(b) \cdot F_X(a).$$

Hence X, Y are independent. $\square$

Example 1: Suppose X and Y have a joint density

$$f(x,y) = 24xy, \quad \text{if } 0<x<1,\ 0<y<1,\ 0<x+y<1.$$

Determine whether X and Y are independent.

<u>Solution</u>: We first calculate the marginal densities $f_X(x)$, $f_Y(y)$.

Notice that

$$f_X(a) = \int_{-\infty}^{-\infty} f(a,y)\, dy$$

$$= \int_0^{1-a} 24ay\, dy$$

$$= 24a \frac{y^2}{2}\Big|_0^{1-a} = 12a\cdot(1-a)^2$$

if $0<a<1$.

$$f_Y(b) = \int_{-\infty}^{\infty} f(x,b)\,dx$$

$$= \int_0^{1-b} 24\,xb\,dx$$

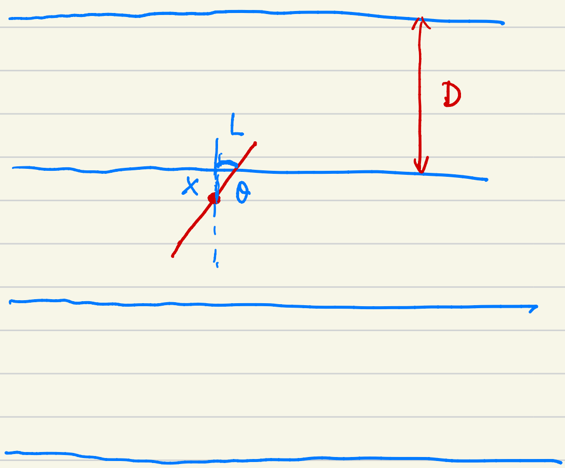
$$= 12b(1-b)^2 \qquad \text{if } 0<b<1.$$

Clearly $f(a,b) \neq f_X(a) f_Y(b)$. Hence

X, Y are not independent.

Example 2. Buffon's needle problem.

A table is ruled with equidistant parallel lines a distance D apart. A needle of length L, where L < D, is randomly thrown on the table. What is the probability that the needle will intersect one of the lines (the other possibility being that the needle will be completely contained in the strip between two lines)?



Solution:

Let X be the distance from the center of the needle to the nearest parallel line.

Let θ be the angle between the verticle line and the needle.

Then the needle intersects a parallel line

$$\Leftrightarrow \quad X \le \tfrac{1}{2} L \cos\theta$$

We may assume that X is unif dist on $[0, \frac{D}{2}]$

Θ is unif dist on $[0, \frac{\pi}{2}]$

and X and Θ are independent

Hence $f_X(x) = \frac{2}{D}$ for $0 < x < \frac{D}{2}$

$f_\Theta(\theta) = \frac{2}{\pi}$ if $0 < \theta < \frac{\pi}{2}$.

Now

$$P\{ X \le \tfrac{L}{2} \cos\theta \}$$

$$= \iint_{\{x \le \frac{L}{2}\cos\theta\}} f_X(x) f_\Theta(\theta) \, dx \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{L}{2}\cos\theta} \frac{4}{D\pi} \, dx \, d\theta$$

$$= \frac{L}{2D\pi}. \quad \square$$