Math 3280 20-10-19

Review.

- Normal r.v X with parameters μ and σ^2:

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad x \in (-\infty, \infty)$$

$Z := \dfrac{X-\mu}{\sigma}$ is a standard normal r.v. (with mean 0, variance 1)

- <u>DeMoivre–Laplace Thm</u>: Given $p \in (0,1)$,

  Let X_n be a binomial rv with parameters (n,p).

Then

$$P\left\{ a < \frac{X_n - np}{\sqrt{np(1-p)}} < b \right\} \longrightarrow \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

as $n \to \infty$.

- Exponential r.v with parameter λ $(\lambda > 0)$,

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Example 1.

Suppose that the length of a phone call in minutes is an exponential random variable with parameter $\lambda = 1/10$. If someone arrives immediately ahead of you at a public telephone booth, find the probability that you will have to wait
(a) more than 10 minutes;
(b) between 10 and 20 minutes.

Solution :

Let X denote the waiting time of the person (in minutes).

According to our assumption, X is an exponential r.v with parameter $\lambda = \frac{1}{10}$.

Hence

(a) $P\{ X \geq 10 \}$

$$= \int_{10}^{\infty} f(x)\, dx = \int_{10}^{\infty} \lambda e^{-\lambda x}\, dx$$

$$= -e^{-\lambda x} \Big|_{10}^{\infty}$$

$$= e^{-\lambda\cdot 10} = e^{-1}.$$

(b) $P\{10 \le X \le 20\}$

$$= \int_{10}^{20} \lambda e^{-\lambda x}\,dx$$

$$= -e^{-\lambda x}\Big|_{10}^{20} = e^{-10\lambda} - e^{-20\lambda}$$

$$= e^{-1} - e^{-2}. \quad \boxtimes$$

§ 5.7 The distribution of a function of a cts r.v.

Q: Let X be a cts r.v. with density f_X.

Let $g: \mathbb{R} \to \mathbb{R}$ be a function.

Let $Y = g(X)$.

How to find the distribution of Y?

Exer 2. Let X be a cts r.v. with density f_X.

Let $Y = X^2$.

Find the pdf of Y.

Solution: We first calculate the CDF of Y:

$$F_Y(y) = P\{Y \le y\}$$

$$= P\{X^2 \le y\}$$

$$= \begin{cases} 0 & \text{if } y < 0 \\ P\{-\sqrt{y} \le X \le \sqrt{y}\} & \text{if } y \ge 0 \end{cases}$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

Taking derivative of F_Y with respect to y gives

$$f_Y(y) = \begin{cases} 0 & \text{if } y \le 0 \\ f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} + f_X(-\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} & \text{if } y > 0 \end{cases}$$

Ex 3. Let X be an exponential r.v with parameter λ. Let $Y = \frac{1}{X}$

Find the pdf of Y.

Solution: First notice that

$$P\{X \leq 0\} = 0$$

Now

$$F_Y(y) = P\{Y \leq y\}$$

$$= P\{\frac{1}{X} \leq y\}$$

$$= \begin{cases} P\{X \geq \frac{1}{y}\} & \text{if } y > 0 \\ 0 & \text{if } y < 0 \end{cases}$$

$$= \begin{cases} 1 - F_X\left(\frac{1}{y}\right) & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Taking derivative of F_Y wrt y gives

$$f_Y(y) = \begin{cases} -f_X\left(\frac{1}{y}\right) \cdot \left(-\frac{1}{y^2}\right) & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} f_X\left(\frac{1}{y}\right) \cdot \frac{1}{y^2} & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \lambda \cdot e^{-\lambda \frac{1}{y}} \cdot \frac{1}{y^2} & \text{if } y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Prop. 4: Let X be a cts r.v. with density f_X. Let $g: \mathbb{R} \to \mathbb{R}$ be a differentiable, strictly monotone function. Let $Y = g(X)$, then the pdf of Y is given by

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \cdot \left| \dfrac{d\, g^{-1}(y)}{dy} \right| & \text{if } y = g(x) \text{ for some } x \\ 0 & \text{otherwise.} \end{cases}$$

pf. Suppose that g is strictly increasing.

Then

$$F_Y(y) = P\{ Y \leq y \}$$

$$= P\{ g(X) \leq y \}$$

$$= \begin{cases} 1 & \text{if } y > \max_{x \in \mathbb{R}} g(x) \\ \boxed{P\{ X \leq g^{-1}(y) \}} & \text{if } y \in \text{Range}(g) \\ 0 & \text{if } y < \min_{x \in \mathbb{R}} g(x) \end{cases}$$

$$P\{ X \leq g^{-1}(y) \}$$

$$= F_X(g^{-1}(y)).$$

Taking derivative of F_Y gives

$$f_Y(y) = \begin{cases} 0 & \text{if } y \notin \text{Rang}(g) \\ f_X(g^{-1}(y)) \cdot \dfrac{d\, g^{-1}(y)}{dy}. \end{cases}$$

if $y = g(x)$

for some x.

$\blacksquare$.