

## Review

- Continuous r.v. :  $\exists$  a non-negative function  $f$  on  $\mathbb{R}$  such that

$$P\{X \in B\} = \int_B f(x) dx, \quad \forall \text{ "measurable" set } B \subset (-\infty, \infty),$$

where "measurable" sets include all intervals and countable unions/intersections of intervals.

In particular,  $P\{a \leq X \leq b\} = \int_a^b f(x) dx.$

and  $\int_{-\infty}^{\infty} f(x) dx = 1.$

We call  $f$  the pdf of  $X$ .

- $E[X] = \int_{-\infty}^{\infty} x f(x) dx.$

Prop: Let  $X$  be a cts r.v. with density  $f$ .

Let  $g: \mathbb{R} \rightarrow \mathbb{R}$ . Then

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx.$$

We prove the above prop only in the case that  $g \geq 0$ .

Lem. Let  $Y$  be a non-negative cts r.v. Then

$$E[Y] = \int_0^{\infty} P\{Y > y\} dy.$$

Proof of the proposition:

By the above lem,

$$E[g(X)] = \int_0^{\infty} P\{g(X) > y\} dy$$

$$= \int_0^{\infty} \left( \int_{\{x: g(x) > y\}} f(x) dx \right) dy$$

$$= \int_0^{\infty} \left( \int_{-\infty}^{\infty} \mathbb{1}_{\{g(x) > y\}} f(x) dx \right) dy$$

$$\left( \text{where } \mathbb{1}_{\{g(x) > y\}} := h(x, y) = \begin{cases} 1 & \text{if } g(x) > y \\ 0 & \text{otherwise} \end{cases} \right)$$

$$\begin{aligned} & \text{(Fubini)} \int_{-\infty}^{\infty} \left( \int_0^{\infty} \mathbb{1}_{\{g(x) > y\}} f(x) dy \right) dx \\ &= \int_{-\infty}^{\infty} f(x) \left( \int_0^{\infty} \mathbb{1}_{\{g(x) > y\}} dy \right) dx \\ &= \int_{-\infty}^{\infty} f(x) \left[ \int_0^{g(x)} 1 dy + \int_{g(x)}^{\infty} 0 dy \right] dx \\ &= \int_{-\infty}^{\infty} f(x) g(x) dx. \end{aligned}$$

□

Def: (Variance) Let  $X$  be a cts r.v. with density  $f$ .

$$V(X) = E[(X - \mu)^2], \quad \text{where } \mu = E[X].$$

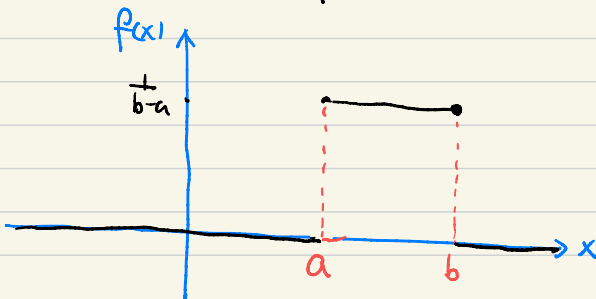
$$\begin{aligned} \bullet \quad V(X) &= \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx \\ &= \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2. \end{aligned}$$

$$\begin{aligned}
 \text{pf. } V(X) &= \int_{-\infty}^{\infty} (x^2 - 2x\mu + \mu^2) f(x) dx \\
 &= \int_{-\infty}^{\infty} x^2 f(x) dx - 2\mu \int_{-\infty}^{\infty} x f(x) dx \\
 &\quad + \mu^2 \int_{-\infty}^{\infty} f(x) dx \\
 &= \int_{-\infty}^{\infty} x^2 f(x) dx - 2\mu \cdot \mu + \mu^2 \\
 &= \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2. \quad \square
 \end{aligned}$$

### § 5.3. Uniform distributions.

Def. A cts  $X$  is said to be uniformly distributed on  $[a, b]$  if it has a density

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$



We call  $X$  is a unif. r.v. on  $[a, b]$ , or say  $X$  has a unif distribution on  $[a, b]$ .

Example 1: Calculate  $E[X]$  and  $V(X)$  for a unif r.v.  $X$  on  $[a, b]$ .

Solution :

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f(x) dx \\ &= \int_a^b x \frac{1}{b-a} dx \\ &= \frac{\frac{1}{2} x^2}{b-a} \Big|_a^b = \frac{\frac{1}{2}(b^2 - a^2)}{b-a} \\ &= \frac{a+b}{2}. \end{aligned}$$

$$\int_{-\infty}^{\infty} x^2 f(x) dx = \int_a^b x^2 \frac{1}{b-a} dx$$

$$= \frac{1}{3} \frac{b^3 - a^3}{(b-a)} = \frac{a^2 + ab + b^2}{3}$$

Hence

$$\begin{aligned} V(X) &= \frac{a^2 + ab + b^2}{3} - \frac{(a+b)^2}{4} \\ &= \frac{(a-b)^2}{12}. \end{aligned}$$

- Def. (Cumulative distribution function).

Let  $X$  be a cts r.v with density  $f$ .

We define

$$\begin{aligned} F_X(b) &= P\{X \leq b\} \\ &= \int_{-\infty}^b f(x) dx. \end{aligned}$$

Prop. If  $f$  is cts at  $b$ , then

$$F_x'(b) = f(b).$$

Pf. Notice that for  $u \in \mathbb{R}$ ,  $u \neq 0$ ,

$$\begin{aligned} \frac{F_x(b+u) - F_x(b)}{u} &= \frac{\int_{-\infty}^{b+u} f(x) dx - \int_{-\infty}^b f(x) dx}{u} \\ &= \frac{1}{u} \int_b^{b+u} f(x) dx \end{aligned}$$

since  $f$  is cts at  $b$ , so

$f(x) - f(b)$  is close to 0

when  $x$  is close to  $b$ ,

hence as  $u \rightarrow 0$ ,  $\frac{1}{u} \int_b^{b+u} f(x) dx \rightarrow f(b)$ .

□.

Example 3.

Let  $X$  be a unif r.v over  $(0, 1)$ .

Find the density of  $X^2$ .

Solution :  $F_{X^2}(b) = P\{X^2 \leq b\}$

$$= \begin{cases} P\{-\sqrt{b} \leq X \leq \sqrt{b}\}, & \text{if } b > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & \text{if } b > 1, \\ \sqrt{b} & \text{if } 0 < b \leq 1, \\ 0 & \text{if } b < 0. \end{cases}$$



Taking derivative we obtain

$$f_{x^2}(b) = \begin{cases} \frac{1}{2\sqrt{b}} & \text{if } b \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

□