Math 3280 20-10-05

Review.

- Bernoulli r.v. with parameter p:

$$P\{X=1\} = p, \quad P\{X=0\} = 1-p.$$

- Binomial r.v. with parameters (n, p):

$$P\{X=i\} = \binom{n}{i} \cdot p^i (1-p)^{n-i},$$

$$i = 0, 1, \cdots, n,$$

Remark: A binomial r.v. X with parameter (n, p) can be written as

$$X = X_1 + \cdots + X_n$$

where $X_1, \cdots, X_n$ are independent Bernoulli r.v.'s.

This follows from the definition of binomial r.v.

§ 4.6 Poisson r.v.

Def. Let $\lambda > 0$. A r.v. X taking values in $\{0, 1, 2, \cdots\}$ is said to be a Poisson r.v. with parameter λ if

$$P\{X = i\} = e^{-\lambda} \cdot \frac{\lambda^i}{i\,!},$$

$$i = 0, 1, \cdots$$

Remark. $$e^{\lambda} = \sum_{i=0}^{\infty} \frac{\lambda^i}{i\,!}.$$

Hence $$\sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i\,!} = 1.$$

A poisson r.v. can be used to approximate a binomial r.v with parameters (n, p) when n is large, p is small so that np is of moderate size.

Let X be a binomial r.v with parameters (n, p).

Let $np = \lambda$.

For $k = 0, 1, \cdots,$

$$P\{X = k\} = \binom{n}{k} p^k \cdot (1-p)^{n-k}$$

$$= \frac{n(n-1)\cdots(n-k+1)}{k!} \cdot \left(\frac{\lambda}{n}\right)^k \cdot \left(1 - \frac{\lambda}{n}\right)^{n-k}.$$

$$= \frac{1 \cdot \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)}{k!} \lambda^k \cdot \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-k}$$

$$\approx \frac{1}{k!} \lambda^k \cdot e^{-\lambda}$$

$\blacksquare$

Expected value and variance of Poisson r.v.

X —— Poisson r.v. with parameter λ.

$$P\{X=k\} = e^{-\lambda} \cdot \frac{\lambda^k}{k!}, \quad k=0,1,\cdots$$

$$E[X] = \sum_{k=0}^{\infty} k \cdot e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= \sum_{k=1}^{\infty} k \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$= \sum_{k=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{(k-1)!}$$

Letting $j=k-1$

$$= \sum_{j=0}^{\infty} e^{-\lambda} \cdot \lambda \cdot \frac{\lambda^j}{j!}$$

$$= \lambda \cdot \sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!}$$

$$= \lambda.$$

$$E[X^2] = \sum_{k=0}^{\infty} k^2 \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$= \sum_{k=1}^{\infty} k \cdot e^{-\lambda} \cdot \frac{\lambda^k}{(k-1)!}$$

$$= \sum_{k=1}^{\infty} (k-1) e^{-\lambda} \cdot \frac{\lambda^k}{(k-1)!}$$

$$+ \sum_{k=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{(k-1)!}$$

$$= \sum_{k=2}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{(k-2)!} + \sum_{k=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{(k-1)!}$$

$$= \lambda^2 + \lambda$$

So $\mathrm{Var}(X) = E[X^2] - E[X]^2$

$$= \lambda^2 + \lambda - \lambda^2$$

$$= \lambda.$$

§ 4.7 Expectation of sums of discrete r.v.'s

Let $X_1, X_2, \cdots, X_n$ be discrete r.v.'s on the same sample space S.

Prop 1. $E[X_1 + \cdots + X_n] = \sum_{k=1}^{n} E[X_k]$.

We will prove the above result under an additional assumption that S is finite or countably infinite.
(The general case is referred to Theoretical Exer 4.36 in the text book).

Lem 2. Assume that S is finite or countably infinite.

Set $p(s) = P(\{s\})$ for $s \in S$.

Then for any r.v. X on S, we have

$$E[X] = \sum_{s \in S} X(s)\, p(s).$$

Pf. Suppose the distinct values of X are x_i, $i \geq 1$.

Let $S_i = \{ s \in S : X(s) = x_i \}$.

Then

$$
\begin{aligned}
E[X] &= \sum_i x_i P\{X = x_i\} \\
&= \sum_i x_i P(S_i) \\
&= \sum_i x_i \sum_{s \in S_i} p(s) \\
&= \sum_i \sum_{s \in S_i} x_i p(s) \\
&= \sum_i \sum_{s \in S_i} X(s) p(s)
\end{aligned}
$$

$$= \sum_{s \in S} X(s)\, p(s)$$

(because $S = \bigcup_i S_i$

with the Union being disjoint) $\square$

<u>Pf of Prop 1.</u>

By Lem 2,

$$E[X_1 + \cdots + X_n] = \sum_{s \in S} \left(X_1(s) + \cdots + X_n(s) \right) p(s)$$

$$= \left(\sum_{s \in S} X_1(s)\, p(s) \right) + \cdots + \left(\sum_{s \in S} X_n(s)\, p(s) \right)$$

$$= E[X_1] + \cdots + E[X_n].$$

$\square$.