Math 3280 20-09-21.

Review.

- Conditional probability

$$P(E|F) = P(EF)/P(F), \quad \text{if } P(F)>0.$$

(conditional prob. of E given F)

- $P(E_1 E_2 \cdots E_n) = P(E_1)\cdot P(E_2|E_1)\cdot P(E_3|E_1E_2)\cdots P(E_n|E_1\cdots E_{n-1})$.

(Multiplicative rule).

- Suppose $F_1, F_2, \cdots, F_n$ are mutually exclusive, and exhausitive (i.e. $\bigcup_{i=1}^{n} F_i = S$).

Then

- $$P(E) = \sum_{i=1}^{n} P(F_i)\cdot P(E|F_i).$$

(law of total probability)

- $$P(F_i|E) = \frac{P(F_i)\,P(E|F_i)}{\sum_{k=1}^{n} P(F_k)\,P(E|F_k)}.$$

(Bayes' formula).

Remark: Suppose $F \subset S$ is an event in a sample space with $P(F) > 0$.

Then $P(\cdot | F)$ is a probability on S.

(1) $P(S|F) = 1$.

(2) $0 \leq P(E|F) \leq 1$.

(3) $P\left(\left(\bigcup_{n=1}^{\infty} E_n\right) \middle| F\right) = \sum_{n=1}^{\infty} P(E_n|F)$,

if $E_1, E_2, \cdots$ are mutually exclusive.

(1), (2) are obvious. To see (3)

$$P\left(\left(\bigcup_{n=1}^{\infty} E_n\right) \middle| F\right) = \frac{P\left(\left(\bigcup_{n=1}^{\infty} E_n\right) \cap F\right)}{P(F)}$$

$$= \frac{P\left(\bigcup_{n=1}^{\infty} (E_n F)\right)}{P(F)}$$

$$= \sum_{n=1}^{\infty} \frac{P(E_n F)}{P(F)} \quad (\text{since } E_n F \text{ are disjoint})$$

$$= \sum_{n=1}^{\infty} P(E_n | F).$$

§ 3.3. Independent events.

Let E, F be two events. In general, knowing that F has occurred changes the chance of E's occurrence, that is,

possibly $P(E|F) \neq P(E)$.

If $P(E|F) = P(E)$, we say E is independent of F.

Notice that

$$P(E|F) = P(E) \iff \frac{P(EF)}{P(F)} = P(E)$$

$$\iff P(EF) = P(E) \cdot P(F)$$

$$\iff P(F|E) = P(F)$$

Def. We say that E and F are independent if

$$\boxed{P(EF) = P(E) \cdot P(F)},$$

Example 1. A card is randomly chosen from a deck of 52 playing cards.

E — the event that the chosen card is an Ace "A"

F — the event that the chosen card is a spade. "♠"

$$P(E) = \frac{4}{52}, \quad P(F) = \frac{13}{52} = \frac{1}{4}.$$

$$P(EF) = \frac{1}{52} = P(E)\,P(F).$$

Hence E, F are independent.

Prop 1. If E and F are independent, then

(1) E and F^c are independent

(2) E^c and F^c are independent.

Pf. (1)

$$P(E \cap F^c) = P(E) - P(EF) \qquad \left(\text{since } E = (E\cap F) \cup (E \cap F^c) \text{ disjoint}\right)$$

$$= P(E) - P(E)\cdot P(F)$$

$$= P(E)\left(1 - P(F)\right)$$

$$= P(E)\,P(F^c),$$

Hence E, F^c are independent.

(2) can be obtained from (1).

- Independence of 3 or more events.

Def. We say 3 events $E, F\ G$ are independent if

(1) $P(EFG) = P(E)P(F)P(G)$.

(2) $P(EF) = P(E)\cdot P(F)$

$P(EG) = P(E)P(G)$

$P(FG) = P(F)P(G)$.

Def. Let $E_1, E_2, \cdots, E_n$ be a finite family of events.

Say $E_1, \cdots, E_n$ are independent if for any sub-collection

$E_{i_1}, E_{i_2}, \cdots, E_{i_r}$ (with $i_1, \cdots, i_r$ being distinct),

$$P(E_{i_1} E_{i_2} \cdots E_{i_r}) = P(E_{i_1}) \cdots P(E_{i_r}).$$

Def: We say an infinite family of events are independent if every finite subfamily of them is independent.