Math 3280 20-09-17

Review

- Sample spaces having equally likely outcomes.

$$P(E) = \frac{\# E}{\# S}.$$

Exer 1.

In the game of bridge, the entire deck of 52 cards is dealt out to 4 players. What is the probability that
(a) one of the players receives all 13 spades;
(b) each player receives 1 ace?

Solution: Let E be the event that one of the players receives all 13 spades.

Let E_i be the event that the i-th player receives all 13 spades, $i=1,2,3,4$.

$E = \bigcup_{i=1}^{4} E_i$, $E_1, \dots, E_4$ are mutually exclusive.

So $P(E) = P(E_1) + P(E_2) + P(E_3) + P(E_4)$.

$$\# E_1 = \binom{39}{13} \cdot \binom{26}{13} \cdot \binom{13}{13}$$

Similarly, $\# E_2 = \# E_3 = \# E_4 = \# E_1$

$$\# S = \binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}$$

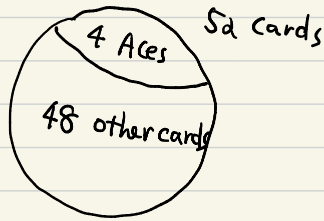
Hence
$$P(E_1) = \frac{\# E_1}{\# S} = \frac{\binom{39}{13}\binom{26}{13}}{\binom{52}{13}\binom{39}{13}\binom{26}{13}} = \frac{1}{\binom{52}{13}}$$

So is $P(E_i)$, $i = 2, 3, 4$.

$$P(E) = \sum_{i=1}^{4} P(E_i) = \frac{4}{\binom{52}{13}}.$$

Let F be the event that each player receives an Ace.

$$\# F = \binom{4}{1}\cdot\binom{48}{12}\cdot\binom{3}{1}\cdot\binom{36}{12}\cdot\binom{2}{1}\binom{24}{12}$$



Hence
$$P(F) = \frac{\binom{4}{1}\binom{48}{12}\binom{3}{1}\binom{36}{12}\binom{2}{1}\binom{24}{12}}{\binom{52}{13}\binom{39}{13}\binom{26}{13}}.$$

Chap 3. Conditional probability and independence

§3.1 Conditional probability.

Example 1: Let us roll two dices. Suppose the first die is a 3. Given this information, what is the prob. that the sum of 2 dices equals 8?

Sol: F — the event that the first die is 3

E — the event that the sum of 2 dices equals 8.

$$F = \{(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)\}$$

$$E = \{(i,j) \in \{1,2,3,4,5,6\}^2 : \quad i+j = 8\}.$$

Prob of each outcome in F is $\frac{1}{6}$.

Hence the (conditional) prob of E given F is $\frac{1}{6}$.

Def. (Conditional prob.)

Let E, F be two events for a random experiment. Suppose $P(F) > 0$. Then the conditional prob. of E given F is

$$P(E|F) = \frac{P(EF)}{P(F)}.$$

Example 2: A coin is flipped twice. What is the conditional prob. that both flips land on heads given that the first flip lands on head?

Sol: Let F be the event that the first flip lands on head. That

$$F = \{(H,H), (H,T)\}.$$

Let E be the event that both flips land on heads.

$$E = \{(H,H)\}.$$

By def, $P(E|F) = \dfrac{P(EF)}{P(F)} = \dfrac{P\{(H,H)\}}{P\{(H,H),(H,T)\}}$

Notice that $S = \{(H,H),(H,T),(T,H),(T,T)\}$

Prop. (Multiplicative rule)

- $P(E_1 E_2) = P(E_1)\, P(E_2 | E_1)$
- $P(E_1 E_2 \cdots E_n)$

$$= P(E_1) \cdot P(E_2|E_1) \cdot P(E_3 | E_1 E_2) \cdots \cdot P(E_n | E_1 E_2 \cdots E_{n-1})$$

Pf. Since $P(E_2|E_1) = \dfrac{P(E_1 E_2)}{P(E_1)}$, so

$$P(E_1 E_2) = P(E_1) \cdot P(E_2|E_1).$$

To see the second identity,

$$\text{RHS} = P(E_1) \cdot \frac{P(E_1 E_2)}{P(E_1)} \cdot \frac{P(E_1 E_2 E_3)}{P(E_1 E_2)} \cdots \frac{P(E_1 \cdots E_n)}{P(E_1 \cdots E_{n-1})}$$

$$= P(E_1 \cdots E_n). \quad \blacksquare$$

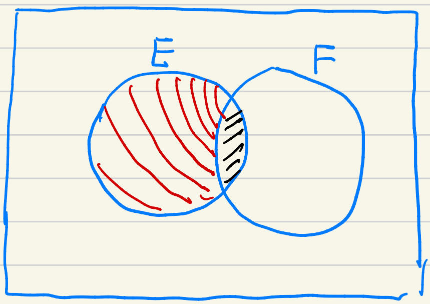
Remark: If $P(F)=0$,

then $P(E|F)$ is not well-defined.

But in practice, you may assign any value from $[0,1]$ to $P(E|F)$.

§ 3.2 Bayes' formula.

Let E, F be two events.



$$E = (E \cap F) \cup (E \cap F^c)$$

(black). (red)

Hence

$$P(E) = P(E \cap F) + P(E \cap F^c)$$

But $$P(E\cap F) = P(F)\cdot P(E|F),$$
$$P(E\cap F^c) = P(F^c)\cdot P(E|F^c).$$

We obtain

$$P(E) = P(F)\cdot P(E|F) + P(F^c)\cdot P(E|F^c).$$

(Total probability formula).

Hence to determine the prob. of E, we may first conduct the "conditioning" upon whether or not the event F has occured.

Next we give a generization of this formula.

Let $F_1, F_2, \cdots, F_n$ be a sequence of events such that they are mutually exclusive, and $\bigcup_{k=1}^{n} F_k = S$ (we say $F_1, \cdots, F_n$ are exhaustive)

Then we have

$$P(E) = \sum_{k=1}^{n} P(F_k) \cdot P(E \mid F_k).$$

pf: Notice that $E = \bigcup_{k=1}^{n} (E \cap F_k)$

(with disjoint union)

Hence

$$P(E) = \sum_{k=1}^{n} P(E \cap F_k)$$

$$= \sum_{k=1}^{n} P(F_k) P(E \mid F_k). \qquad \square$$

Prop. (Bayes' formula).

Assume $F_1, \cdots, F_n$ are mutually exclusive and exhaustive.

Then for any $1 \leq i \leq n$,

$$P(F_i \mid E) = \frac{P(F_i) \cdot P(E \mid F_i)}{\sum_{k=1}^{n} P(F_k) P(E \mid F_k)}$$

Pf:

$$\sum_{k=1}^{n} P(F_k)\,P(E|F_k) = P(E)$$

$$P(F_i)\cdot P(E|F_i) = P(EF_i)$$

$\blacksquare$

Example 3.

A bin contains 3 types of disposable flashlights. The probability that a type 1 flashlight will give more than 100 hours of use is .7, with the corresponding probabilities for type 2 and type 3 flashlights being .4 and .3, respectively. Suppose that 20 percent of the flashlights in the bin are type 1, 30 percent are type 2, and 50 percent are type 3.

(a) What is the probability that a randomly chosen flashlight will give more than 100 hours of use?
(b) Given that a flashlight lasted more than 100 hours, what is the conditional probability that it was a type j flashlight, j = 1, 2, 3?

Solution: E —— the event that a random chosen flashlight will give more than 100 hours.

F_i $(i=1,2,3)$

—— the event that a random chosen flashlight is of type i.

We need to find out (a) $P(E)$

(b) $P(F_i | E)$.

From the conditions of the question, we know

$$P(E|F_1) = 0.7, \quad P(E|F_2) = 0.4$$

$$P(E|F_3) = 0.3.$$

$$P(F_1) = 0.2, \quad P(F_2) = 0.3, \quad P(F_3) = 0.5.$$

Hence $$P(E) = \sum_{i=1}^{3} P(F_i)\, P(E|F_i)$$

$$= 0.2 \times 0.7 + 0.3 \times 0.4 + 0.5 \times 0.3$$

$$P(F_1|E) = \frac{P(F_1) \cdot P(E|F_1)}{P(E)}$$

$$= \frac{0.2 \times 0.7}{0.2 \times 0.7 + 0.3 \times 0.4 + 0.5 \times 0.3}.$$

$$= \frac{14}{41}$$

Similarly

$$P(F_2|E) = \frac{12}{41}, \quad P(F_3|E) = \frac{15}{41}. \quad \boxed{/\!/}$$