Introductory Probability

20-09-10.

Review.

Random experiment; outcomes.

Sample space; event.

Operations on events : union, intersection, complement.

§2.3. Axioms of probability.

Q: How can we define the prob. of an event ?

An intuitive approach:

repeat the random experiment n times.

Let $n(E)$ be the times that an event E occurs

Let $$P(E) = \lim_{n\to\infty} \frac{n(E)}{n}.$$

Drawbacks: ① why does the limit exist?

② Even if the limit exist, why is it independent of the experiments?

The axiomatic approach to prob. (by Kolmogrov)

Def. (Prob. of an event).

Let S be the sample space of a random experiment.

A probability P on S is a function that assigns a value to each event E such that the following 3 axioms hold:

Axiom 1: $0 \leq P(E) \leq 1$, $\forall$ event E.

Axiom 2: $P(S) = 1$.

Axiom 3: If $E_1, E_2, \ldots,$ are a sequence of events which are mutually exclusive,

then

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n).$$

(Countable additivity of prob.)

§ 2.4. Some properties of probability.

Prop 1. $P(\emptyset) = 0$.

Pf. Let $E_1 = S$, and $E_n = \emptyset$ for $n = 2, 3, \cdots$.

Then $E_1, E_2, \cdots,$ are mutually exclusive.

By Axiom 3,

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n)$$

$$= P(E_1) + P(E_2) + \cdots$$

$$= P(S) + P(\emptyset) + P(\emptyset) + \cdots$$

LHS ≤ 1, RHS ≤ 1 only occurs when $P(\emptyset) = 0$. $\square$

Prop 2. $P(E^c) = 1 - P(E)$.

Pf. Notice that

$$S = E^c \cup E \cup \phi \cup \phi \cdots$$

By Axiom 3 and Prop 1,
Axiom 2

$$1 = P(S) = P(E^c) + P(E).$$

$\square$

Prop 3. Let E, F be two events. Then

$$P(E \cup F) = P(E) + P(F) - P(E \cap F).$$

Pf. $$E \cup F = E \cup (F \setminus E)$$

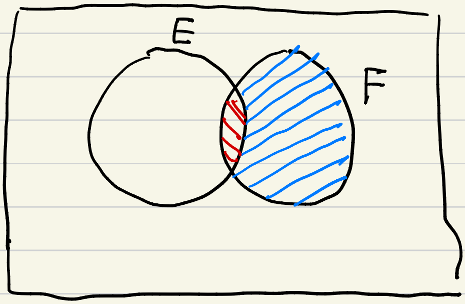
since $E \cap (F \setminus E) = \phi$, so by Axiom 3,

$$P(E \cup F) = P(E) + P(F \setminus E). \quad ①$$

Now we consider $P(F\setminus E)$.

Notice that

$$F = (F\setminus E) \cup (E\cap F)$$



red $\leftrightarrow$ $E\cap F$

blue $\leftrightarrow$ $F\setminus E$.

Using Axiom 3 again,

$$P(F) = P(F\setminus E) + P(E\cap F)$$

hence

$$P(F\setminus E) = P(F) - P(E\cap F) \quad ②$$

Plugging ② into ① yields the desired identity. $\boxed{///}$

Prop 4. (Inclusion-exclusion identity).

$$P(E_1 \cup \cdots \cup E_n) = \sum_{i=1}^{n} P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} \cap E_{i_2})$$

$$+ \sum_{i_1 < i_2 < i_3} P(E_{i_1} E_{i_2} E_{i_3}) - \cdots + (-1)^{n+1} P(E_1 E_2 \cdots E_n)$$

$$= \sum_{r=1}^{n} (-1)^{r+1} \cdot \sum_{i_1 < \cdots < i_r} P(E_{i_1} E_{i_2} \cdots E_{i_r})$$

Pf. By induction on n.

By prop 3, the identitity holds for $n=2$.

Next suppose the identity holds for $n=k$.

Then

$$P(E_1 \cup \cdots \cup E_k \cup E_{k+1})$$

$$= P((E_1 \cup \cdots \cup E_k) \cup E_{k+1})$$

$$= P(E_1 \cup \cdots \cup E_k) + P(E_{k+1})$$

$$- P\big((E_1 E_{k+1}) \cup (E_2 E_{k+1}) \cdots \cup (E_k E_{k+1})\big)$$

Using induction on $n=k$

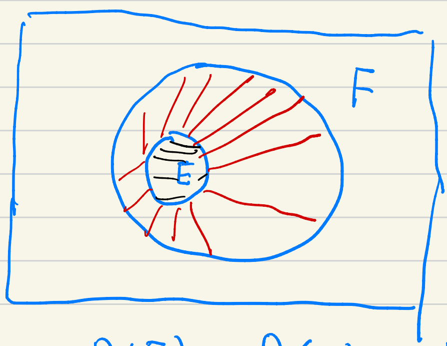
$= \ldots\ldots$

$=$ desired sum. $\square$

Prop 5. Suppose $E \subset F$. Then

$$P(E) \leq P(F).$$

Pf. $F = E \cup (F \setminus E)$.



By Axiom 3, $P(F) = P(E) + P(F \setminus E)$

Notice by Axiom 1, $P(F \setminus E) \geq 0$,

so $P(F) \geq P(E)$.

$\boxtimes$.

Prop 6. Let $E_1, E_2, \cdots,$ be a sequence of events.

Then

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} P(E_n).$$

(Countable sub-additivity of prob.)

Proof. First we write $\bigcup_{n=1}^{\infty} E_n$ as the union of some disjoint events. To do so, write

$$F_1 = E_1$$

$$F_2 = E_2 \setminus E_1$$

$$F_3 = E_3 \setminus (E_1 \cup E_2),$$

$\cdots\cdots$

$$F_n = E_n \setminus \left(\bigcup_{i=1}^{n-1} E_i \right),$$

$\cdots\cdots$

Then

- $F_n \subset E_n, \quad n = 1, \cdots,$
- $\bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} E_i \qquad (*)$
- $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i$
- $F_1, F_2, \cdots$ are mutually exclusive. ✓

To show $(*)$, recall that $F_i \subset E_i$ so

$$\bigcup_{i=1}^{n} F_i \subset \bigcup_{i=1}^{n} E_i.$$

To prove $\bigcup_{i=1}^{n} F_i \supset \bigcup_{i=1}^{n} E_i$,

let $x \in \bigcup_{i=1}^{n} E_i$. Then $x \in E_i$ for some $i \leq n$.

Let i be the smallest integer $\leq n$ such that

$$x \in E_i$$

Then
$$x \in E_i \Big\backslash \bigcup_{j=1}^{i-1} E_j = F_i$$

which means

$$\bigcup_{i=1}^{n} E_i \subset \bigcup_{i=1}^{n} F_i ,$$

which proves (*)

Now using Axiom 3 to $P\left(\bigcup_{n=1}^{\infty} F_n\right)$

we have

$$P\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} P(F_n)$$

$$\leq \sum_{n=1}^{\infty} P(E_n),$$

and we are done since

$$P\left(\bigcup_{n=1}^{\infty} F_n\right) = P\left(\bigcup_{n=1}^{\infty} E_n\right). \quad \blacksquare$$

Prop 7 (Continuity of prob.)

(i) Assume $E_1 \subset E_2 \subset \cdots$ is an inceasing sequence of events.

Then $$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} P(E_n)$$

(ii) Suppose $E_1 \supset E_2 \supset E_3 \cdots$ is a decreasing sequence of events. Then

$$P\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} P(E_n).$$

Pf. We first prove (i).

Define $F_1 = E_1$,

$F_2 = E_2 \setminus E_1$,

$\cdots\cdots$

$F_n = E_n \setminus (E_1 \cup \cdots \cup E_{n-1})$

$\cdots\cdots$

Then as was proved in the last proposition,

(F_n) are mutually exclusive

and
$$\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n$$

$$\bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} E_i = E_n$$

Hence

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = P\left(\bigcup_{n=1}^{\infty} F_n\right)$$
$$= \sum_{n=1}^{\infty} P(F_n)$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} P(F_i)$$
$$= \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} F_i\right)$$
$$= \lim_{n\to\infty} P(E_n).$$

Next we derive (ii) from (i).

Suppose $E_1 \supset \cdots \supset E_n \cdots$ is a decreasing sequence of events

Then $E_1^c \subset E_2^c \subset \cdots$

By (i)

$$P\left(\bigcup_{n=1}^{\infty} E_n^c\right) = \lim_{n\to\infty} P(E_n^c).$$

Notice that by De Morgan's law,

$$\left(\bigcup_{n=1}^{\infty} E_n^c\right)^c = \bigcap_{n=1}^{\infty} E_n$$

$$(E_n^c)^c = E_n$$

$$P\left(\bigcup_{n=1}^{\infty} E_n^c\right) = 1 - P\left(\bigcap_{n=1}^{\infty} E_n\right)$$

$$P(E_n^c) = 1 - P(E_n).$$

Hence

$$1 - P\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty}\left(1 - P(E_n)\right).$$

So

$$P\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} P(E_n).$$

$\blacksquare$.