The Chinese University of Hong Kong
MATH3280 Introductory Probability 2020-2021
Final Examination

December 14 9:30am–12:30pm (noon)

Instructions

- This is an open-book examination. You may **ONLY** refer to printed/written materials during the examination. Assessing information on the internet is not allowed.
- You are allowed to use a calculator in the approved list during the examination.
- You shall take the examination in isolation and shall not communicate with any person during the examination other than the course teacher(s) concerned. Please kindly be reminded of the following regulations enforced by the university:

  Honesty in Academic Work: *The Chinese University of Hong Kong places very high importance on honesty in academic work submitted by students, and adopts a policy of zero tolerance on cheating and plagiarism. Any related offence will lead to disciplinary action including termination of studies at the University.*
- There are a total of 100 points.
- Answer **ALL** questions.
- Show all steps clearly in your working. **NO** point will be given if suffcient justification is not provided.
- Only handwritten answers on papers or electronic devices will be accepted. Typed answer will **NOT** be accepted.
- Please follow the instruction of submission below:
  1. Write your answers on papers or electronic devices. Only handwritten answers will be accepted and typed answer will **NOT** be accepted.
  2. Scan or take photos of your work (if you write on papers).
  3. Combine your work into a single pdf file.
  4. Name the pdf file by your student id (e.g. 1155123456.pdf).
  5. You must upload the pdf file to Blackboard before 12:30pm (noon), 14 Dec, 2020. Mark deduction will be made for late submission.
  6. Please check your file carefully to make sure no missing pages.

1 (10pts). Let X be a continuous random variable, having a density function given by

$$f(x) = \begin{cases} c(x^2 - x^3), & \text{if } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(i) What is the value of c?

(ii) Compute $E[X]$.

(iii) Compute $\text{Var}[X]$.

2 (15pts). A random variable Z is uniformly distributed on $\{1, 2, \ldots, 10\}$. Let X and Y be two random variables defined by

$$X = \begin{cases} 1 & \text{if } Z \geq 5 \\ 0 & \text{otherwise} \end{cases}, \qquad Y = \begin{cases} 1 & \text{if } Z \text{ is even} \\ 0 & \text{otherwise} \end{cases}.$$

(a) Find $E[X]$ and $E[Y]$.

(b) Are X and Y independent?

(c) Find $\text{Cov}(X, Y)$.

3 (10pts) In the World Series, two teams play a series of games, and the first team to win five games wins the series. Suppose that each game ends in either a win or a loss for your team, and that for each game that is played the chance of a win for your team is p, independently of what happens in other games. What is the probability that your team wins the series?

4 (10pts). Five distinct numbers are randomly distributed to players numbered 1 through 5. Whenever two players compare their numbers, the one with the higher one is declared the winner. Initially, players 1 and 2 compare their numbers; the winner then compares his number with that of player 3, and so on. Let X denote the number of times player 1 is a winner. Find $P\{X = i\}$, $i = 0, 1, 2, 3, 4$.

——————(Please Turn Over)——————

5 (15pts). The joint density function of X and Y is given by

$$f(x,y) = \begin{cases} \dfrac{4y}{x} & \text{if } 0 < x < 1 \text{ and } 0 < y < x, \\ 0 & \text{otherwise.} \end{cases}$$

Find the following:

(i) $E[XY]$.

(ii) The marginal density of X.

(iii) $E[Y|X = x]$.

(iv) The density of $X + Y$.

6 (10pts) Let X be a non-negative continuous random variable with density function f.

(a) Show that $E[X] = \int_0^\infty P\{X > t\}\, dt$.

(b) Show that $(E[X])^2 < E[X^2]$.

7 (15pts) Let X, Y be independent standard normal random variables. Find the probability density functions of each of the following random variables:

(i) X^2;

(ii) $X^2 + Y^2$;

(iii) $X + Y$.

8 (15pts) Let U, V be two independent uniform random variables on the interval $(0, 1)$. Let

$$X = \min(U, V), \quad Y = \max(U, V),$$

where $\min(u, v)$ is the smaller and $\max(u, v)$ the larger of two numbers u and v.

(a) Find the probability density function f_X of X.

(b) Find the probability joint density function $f_{X,Y}$ of X, Y.

(c) Find $P\{X \leq 1/3 | Y \geq 1/2\}$.

——————— End ———————