(b) Is V a vector subspace of $\mathrm{Mat}_{2\times 2}(\mathbb{C})$?

3. Let V be a vector space over $\mathbb{R}$, and W_1, W_2 be vector subspaces of V.

   (a) Show by example that $W_1 \cup W_2$ may not be a vector subspace of V.

   (b) Show that if $W_1 \cup W_2$ is a vector subspace, then either $W_1 \subset W_2$, or $W_2 \subset W_1$

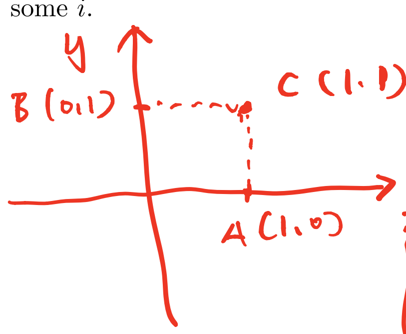
   (c) If

$$V = \bigcup_{i=1}^{n} U_i$$

$n = 3$. $U_1 \quad U_2 \quad U_3$

   for some vector subspaces $U_1, \ldots, U_n$ of V, show that $U_i = V$ for some i.

R^2 plane.



x-axis: $R \times \{0\}$

y-axis: $\{0\} \times R$

$\boxed{\text{Check } x \text{ and } y\text{-axis are both subspaces of } R^2}$

A: $(1,0) \in R \times \{0\}$

B: $(0,1) \in \{0\} \times R$

$C = A + B = (1,1)$

Q: Does C belong x-axis or y-axis?

$C \notin x \cup y$

(b) Here, we assume there exists

$w_1 \in W_1 \setminus W_2$ $\qquad$ $w_2 \in W_2 \setminus W_1$

$w_3 = w_1 + w_2 \in W_1 \cup W_2$

Next. ① $\underline{w_3} \in \underline{\underline{W_1}}$

$$-\underline{w_1} + w_3 \in W_1$$

$$= -w_1 + (w_1 + w_2)$$

$$= w_2 \in W_1$$

which contradicts our assumption.

② $w_3 \in W_2$

$$-w_2 + w_3 \in W_2$$

$$= -w_2 + (w_1 + w_2)$$

$$= w_1 \in W_2 \qquad \times$$

$$W_1 \subset W_2 \quad \text{or} \quad W_2 \subset W_1$$

2 Problems

1. In the vector space Mat $_{2\times 2}(\mathbb{C})$, determine whether the following statements are correct.

(a) The matrix $\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$ is in the span of $\left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$

(b) The matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is in the span of $\left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$

a is correct.

(a) $\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} = a\cdot\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + b\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$

if there isn't any solution for the above equation. then.

$$\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} a+b & b+c \\ -a & c \end{pmatrix} \Leftrightarrow \begin{cases} a+b=1 \\ b+c=2 \\ -a=-3 \\ c=4 \end{cases} \Rightarrow \begin{cases} a=3 \\ b=-2 \\ c=4 \end{cases}$$

(b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + b\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} a+b & b+c \\ -a & c \end{pmatrix} \Rightarrow \begin{cases} a+b=1 \\ b+c=0 \\ -a=0 \\ c=1 \end{cases} \Rightarrow \cancel{\begin{cases} a=0 \\ b=1 \\ c=1 \end{cases}}$$

b is not correct

2. Determine whether the following sets are linearly independent.

(a) $\left\{\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -2 & 6 \\ 4 & -5 \end{pmatrix}\right\}$ in $\mathrm{Mat}_{2\times 2}(\mathbb{R})$

(b) $\{x^3 - 2x^2, -x^2 + 3x - 1, (x-1)^3\}$ in $P_3(\mathbb{R})$

(a). $a, b \in \underline{F}$ (Real field. $\mathbb{R}$) (why?).

☆ $a\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix} + b\begin{pmatrix} -2 & 6 \\ 4 & -5 \end{pmatrix} = 0$

Check. whether a and b are both 0?

① if. $\underline{a\ b \neq 0}$ a, b are linealy dependent.

② if $a = b = 0$, a, b linealy independent.

☆: $\begin{pmatrix} a-2b & -3a+6b \\ -2a+4b & 4a-5b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \boxed{0} \Rightarrow \begin{cases} a-2b=0 \\ 4a-5b=0 \end{cases}$

$\Rightarrow \begin{cases} a=0 \\ b=0 \end{cases}$

#. linearly independent!

(b) $a(x^3 - 2x^2) + b(-x^2 + 3x - 1) + c(x^3 - 3x^2 + 3x - 1) = 0$

$\Rightarrow (a+c)x^3 - (2a+3c)x^2 + (3b+3c)x - (b+c) = 0$

$\begin{cases} a+c=0 \\ 2a+3c=0 \\ 3b+3c=0 \\ b+c=0 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=0 \\ c=0 \end{cases}$ # linearly independent!