Week 5

MATH 2040B

October 13, 2020

1 Concepts

1. Matrix representation: Given a linear transformation T from finite dimension vector space U to finite dimension vector space V, $\beta = \{\beta_1, \cdots, \beta_n\}$ is an ordered basis of U and $\gamma = \{\gamma_1, \cdots, \gamma_m\}$ is an ordered basis of V, if $T(\beta_j) = \sum_{i=1}^{m} T_{ij}\gamma_i$, then the $m \times n$ matrix $[T]_\beta^\gamma = (T_{ij})$ is called the matrix representation of T.

2. The coordinate representation corresponing to γ of $T(\beta_j)$ is $[T(\beta_j)]_\gamma = (T_{1j}, \cdots, T_{mj})^T$, and $[T]_\beta^\gamma$ can be writed as the combination of these column vectors $[T]_\beta^\gamma = [[T(\beta_1)]_\gamma, \cdots, [T(\beta_n)]_\gamma]$.

3. If $U = V$ and $\beta = \gamma$, then the matrix representation of T can be simplified as $[T]_\beta$

2 Problems

1. $T : \mathbb{R}^2 \to \mathbb{R}^3$, $T(a, b) = (a - b, a + b, a)$, find $[T]_\beta^\gamma$ where β, γ are standard basis of $\mathbb{R}^2$ and $\mathbb{R}^3$.

2. $T : \mathbb{R}^2 \to \mathbb{R}^3$, $T(a,b) = (a-b, a+b, a)$,find $[T]_\beta^\gamma$ where $\beta = \{(1,1),(1,-1)\}$ and $\gamma = \{(0,1,1),(1,0,1),(1,1,0)\}$.

3. Let $\beta = \{\beta_1, \cdots, \beta_n\}$, $\gamma = \{\gamma_1, \cdots, \gamma_m\}$ be basis of vector space U and V. $\mathcal{L}(U, V)$ is the set of all linear transformation from U to V. It is known that $\mathcal{L}(U, V)$ is a vector space with addition: $(P + Q)(u) = P(u) + Q(u)$ and scalar multiple: $(aP)(u) = aP(u)$. Show that the map $T : \mathcal{L}(U, V) \to M_{m\times n}(F)$ that $T(P) = [P]_\beta^\gamma$ is linear.

4. Prove that $T : \mathcal{L}(U, U) \to M_{n\times n}(F)$ is injective.

5. Suppose $P : U \to U$ is linear, show that the following two statements are equivalent.

(1). $P^2 = P$;

(2). For some basis η of U and $r \leq n = \dim(U)$,

$$[P]_\eta = \begin{pmatrix} I_{r\times r} & 0 \\ 0 & 0 \end{pmatrix}$$