Week 12

MATH 2040

December 3, 2020

1 Problems

1. Let T and U be self-adjoint operators on an inner product space V. Prove that TU is self-adjoint if and only if $TU = UT$.

   Ans: $(TU)^* = U^*T^* = UT$, so $TU = UT$ is equivalent to $(TU)^* = TU$, then TU is self-adjoint.

2. Let T be a self-adjoint operator on a finite-dimensional inner product space V. Prove that

   (a) For all $x \in V$, $\|T(x) \pm ix\|^2 = \|T(x)\|^2 + \|x\|^2$.

   (b) $T - iI$ is invertible and $[(T - iI)^{-1}]^* = (T + iI)^{-1}$

   Ans:

   (a) Since T is self-adjoint, we have $T^* = T$, then $\langle x, T(x)\rangle = \langle x, T^*(x)\rangle = \langle T(x), x\rangle$. Therefore,

$$\begin{aligned}\|T(x) \pm ix\|^2 &= \|T(x)\|^2 + \|x\|^2 \pm \langle T(x), ix\rangle \pm \langle ix, T(x)\rangle \\ &= \|T(x)\|^2 + \|x\|^2 \mp i\langle T(x), x\rangle \pm i\langle x, T(x)\rangle \\ &= \|T(x)\|^2 + \|x\|^2 \mp i\langle T(x), x\rangle \pm i\langle T(x), x\rangle \\ &= \|T(x)\|^2 + \|x\|^2\end{aligned}$$

   (b) Suppose $(T \pm iI)(x) = 0$, then $\|(T \pm iI)(x)\|^2 = \|T(x) \pm ix\|^2 = \|T(x)\|^2 + \|x\|^2 = 0$, which means $x = 0$, so $T \pm iI$ is surjective and then invertible.
   Since T is self-adjoint, we have that $(T - iI)^* = T^* + iI = T + iI$, then,

$$\begin{aligned}\langle x, [(T - iI)^{-1}]^*(T + iI)(y)\rangle &= \langle (T - iI)^{-1}(x), (T + iI)(y)\rangle \\ &= \langle (T - iI)^{-1}(x), (T - iI)^*(y)\rangle \\ &= \langle (T - iI)(T - iI)^{-1}(x), y\rangle \\ &= \langle x, y\rangle.\end{aligned}$$

   This shows that $[(T - iI)^{-1}]^*(T + iI) = I$ and so $[(T - iI)^{-1}]^* = (T + iI)^{-1}$.

3. Let T be a normal operator on a finite-dimensional real inner product space V whose characteristic polynomial splits. Prove that V has an orthonormal basis of eigenvectors of T. Hence prove that T is self-adjoint.

   Ans: Please find the proof in lecture note.

4. Prove $\det(A^*) = \overline{\det(A)}$, $A \in M_{n\times n}(\mathbb{C})$.

Ans: We know that $A^* = \overline{A^t}$ and $\det(A^t) = \det(A)$, so it's sufficient to prove $\det(\bar{A}) = \overline{\det(A)}$.

Claims that $\det(A^*) = \overline{\det(A)}$ holds for all $M_{n\times n}(\mathbb{C})$.

For $n = 1$, it's clear that $\det(\bar{A}) = \overline{\det(A)}$.

When $n > 1$, suppose claim holds for $n - 1$, then

$$\overline{\det(A)} = \overline{\sum_{j=1}^{n}(-1)^{i+j} A_{ij} \det(\tilde{A}_{ij})} = \sum_{j=1}^{n}(-1)^{i+j}\overline{A_{ij}}\overline{\det(\tilde{A}_{ij})},$$

where i is arbitrary, A_{ij} is the i, j element of matrix A and $\tilde{A}_{ij}$ is the i, j minor of A. Since $\tilde{A}_{ij} \in M_{n-1\times n-1}(\mathbb{C})$, then $\overline{\det(\tilde{A}_{ij})} = \det(\bar{\tilde{A}}_{ij})$ and $\bar{A}_{ij} = \overline{A_{ij}}$, so

$$\det(\bar{A}) = \sum_{j=1}^{n}(-1)^{i+j} \bar{A}_{ij} \det(\bar{\tilde{A}}_{ij}) = \sum_{j=1}^{n}(-1)^{i+j}\overline{A_{ij}}\overline{\det(\tilde{A}_{ij})} = \overline{\det(A)},$$

claim holds for n. By M.I. we have $\det(A^*) = \overline{\det(A)}$.

5. Let $T : V \to W$ be a linear transformation, where V and W are finite dimensional inner product spaces with inner products $\langle , \rangle_1$ and $\langle , \rangle_2$, respectively. Prove the following results.

   (a) There is a unique adjoint T^* of T, and T^* is linear.

   (b) If β and γ are orthonormal bases for V and W, respectively, then $[T^*]_\gamma^\beta = ([T]_\beta^\gamma)^*$

   (c) $\text{rank}T^* = \text{rank}T$

   (d) $\langle T^*(x), y\rangle_1 = \langle x, T(y)\rangle_2$ for all $x \in W$ and $y \in V$.

   (e) For all $x \in V$, $T^*T(x) = 0$ if and only if $T(x) = 0$.

   Ans:

   (a) For some fixed $y \in W$, define $g_y(x) = \langle T(x), y\rangle_2$. It's easy to check g_y is linear that $\forall x, z \in V$ and $\forall c \in \mathbb{C}$,

   $$g_y(x+cz) = \langle T(x+cz), y\rangle_2 = \langle T(x)+cT(z), y\rangle_2 = \langle T(x), y\rangle_2+c\langle T(z), y\rangle_2 = g_y(x)+cg_y(z)$$

   So we can find a unique $\tilde{y} \in V$ such that $g_y(x) = \langle x, \tilde{y}\rangle_1$ for any $x \in V$, so we can define $T^*(y) = y$ then $\langle x, T^*(y)\rangle_1 = g_y(x) = \langle T(x), y\rangle_2$. Since $\tilde{y}$ can be uniquely determined by y, such T^* exists and is well-defined. Then we can also check that for $\forall x \in V$, $\forall y, z \in W$ and $\forall c \in \mathbb{C}$,

   $$\begin{aligned}\langle x, T^*(y + cz)\rangle_1 &= \langle T(x), y + cz\rangle_2 = \langle T(x), y\rangle_2 + \bar{c}\langle T(x), z\rangle_2 \\ &= \langle x, T^*(y)\rangle_1 + \bar{c}\langle x, T^*(z)\rangle_1 = \langle x, T^*(y) + cT^*(z)\rangle_1,\end{aligned}$$

   so we have $T^*(y + cz) = T^*(y) + cT^*(z)$, T^* is linear.

   (b) Let $\beta = \{u_1, \cdots, u_n\}$ and $\gamma = \{v_1, \cdots, v_m\}$, assume that $T(u_i) = \sum_{j=1}^m a_{ij}v_j$ and $T^*(v_i) = \sum_{j=1}^n b_{ij}u_j$, then $[T]_\beta^\gamma = (a_{ij})$ and $[T^*]_\gamma^\beta = (b_{ij})$. For $\overline{a_{ij}}$,

   $$\overline{a_{ij}} = \overline{\langle T(u_i), v_j\rangle_2} = \overline{\langle u_i, T^*(v_j)\rangle_1} = \langle T^*(v_j), u_i\rangle = b_{ji},$$

   so $[T^*]_\gamma^\beta = ([T]_\beta^\gamma)^*$

   (c) in HW10.

   (d) $\langle T^*(x), y\rangle_1 = \overline{\langle y, T^*(x)\rangle_1} = \overline{\langle T(y), x\rangle_2} = \langle x, T(y)\rangle_2$

   (e) $\Rightarrow$: If $T^*T(x) = 0$, then $\|T(x)\|^2 = \langle T(x), T(x)\rangle_2 = \langle T^*T(x), x\rangle_1 = \langle 0, x\rangle_1 = 0$, so $T(x) = 0$.
   $\Leftarrow$: If $T(x) = 0$ then $T^*T(x) = T^*(0) = 0$.