Matrix representation

Notation: An ordered basis for a finite-dimensional vector space V is a basis for V endowed with a specific order.

(e.g. $\mathbb{R}^2$ $\quad \underbrace{\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}}_{= \beta_1} \neq \underbrace{\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}}_{= \beta_2}$ as ordered basis)

Definition: Let V be a finite-dimensional vector space and $\beta = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ be an ordered basis for V.

Then, $\forall \vec{x} \in V$, $\exists !\ a_1, a_2, \dots, a_n \in F$ s.t. $\vec{x} = \sum_{i=1}^{n} a_i \vec{u}_i$.

The coordinate vector of $\vec{x}$ relative to β, denoted as $[\vec{x}]_\beta$, is the column vector $\underset{(F^n)}{}$

$$[\vec{x}]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in F^n$$

Remark: Define a map $V \to F^n$. This map is linear
$$\vec{x} \mapsto [\vec{x}]_\beta$$
(HW. $[\underbrace{a\vec{x}+\vec{y}}_{\in V}]_\beta = a[\vec{x}]_\beta + [\vec{y}]_\beta$)

Now, suppose V and W are finite-dimensional vector spaces with ordered bases $\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ (for V) and $\gamma = \{\vec{w}_1, \dots, \vec{w}_m\}$ (for W) respectively.

Let $T: V \to W$ be a linear transformation.

Then for each $1 \le j \le n$, $\exists\, a_{ij} \in F$ $1 \le i \le m$ such that

$$\underbrace{T(\vec{v}_j)}_{\in W} = \sum_{i=1}^{m} a_{ij}\vec{w}_i \quad \text{for} \quad 1 \le j \le n.$$

Definition: With this notation as above, we call the matrix

$$A :\overset{\text{def}}{=} (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

the matrix representation of T in the ordered bases β and γ, and denoted it as $A = [T]_\beta^\gamma$.

If $V = W$ and $\beta = \gamma$, then we simply write $\cancel{[T]_\beta^\beta}$ $[T]_\beta$

$$T(\vec{v}_j) = \sum_{i=1}^{m} a_{ij} \vec{w}_i \quad \text{for} \quad 1 \le j \le n,$$

$\hat{W}$

$$T(\vec{v}_j) = \sum_{i=1}^{m} a_{ij}\,\vec{w}_i \quad \text{for} \quad 1 \le j \le n,$$

$$\in W$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ a_{31} & a_{32} & \vdots \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{mn} \end{pmatrix}$$

$$\begin{matrix} \| & \| & \| \\ [T(\vec{v}_1)]_\gamma & [T(\vec{v}_2)]_\gamma & [T(\vec{v}_n)]_\gamma \end{matrix}$$

$$T(\vec{v}_1) = \sum_{i=1}^{m} a_{i1}\,\vec{w}_i$$

$$\in W$$

$$\Downarrow$$

$$[T(\vec{v}_1)]_\gamma = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \in F^n$$

$\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ for V

$\gamma = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m\}$ for W

$$[T]_\beta^\gamma = \begin{array}{c} \\ m \\ \\ \end{array} \left[\underbrace{\begin{pmatrix} | & | & & | \\ [T(\vec{v}_1)]_\gamma & [T(\vec{v}_2)]_\gamma & \cdots & [T(\vec{v}_n)]_\gamma \\ | & | & & | \end{pmatrix}}_{n} \right]$$

$\in$

$M_{m \times n}$

Examples:

- Let $A \in M_{m\times n}(F)$. $L_A : F^n \to F^m$ defined by : $L_A(\underset{\in F^n}{\vec{x}}) \stackrel{\text{def}}{=} A\vec{x}$

Let β and γ be the standard bases for F^n and F^m resp.

$$\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \dots \right\}$$

$$[L_A]_\beta^\gamma = \begin{pmatrix} | & & | \\ [A\vec{e}_1]_\gamma & \dots & [A\vec{e}_n]_\gamma \\ | & & | \end{pmatrix}$$

first col. of A

n^{th} col of A

$$A\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \text{first col of } A$$

$$= \begin{pmatrix} | & & | \\ 1^{st} \text{ col of } A & & n^{th} \text{ col of } A \\ | & & | \end{pmatrix} = A$$

• For $T: P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ defined as $T(f(x)) = f'(x)$.

Let $\beta = \{1, x, x^2, \dots, x^n\}$ be an ordered basis for $P_n(\mathbb{R})$

Let $\gamma = \{1, x, x^2, \dots, x^{n-1}\}$ be an ordered basis for $P_{n-1}(\mathbb{R})$

$$[T]_\beta^\gamma = \begin{pmatrix} | & | & & | \\ [\underbrace{T(1)}_{=0}]_\gamma & [\overbrace{T(x)}^{=1}]_\gamma & \cdots & [\overbrace{T(x^n)}^{=nx^{n-1}}]_\gamma \\ | & | & & | \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & n \end{pmatrix}$$

Lecture 10:

Recall: • Coordinate representation of $\vec{v} \in V$ w.r.t. $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$ ordered

Write $\vec{v} = a_1 \vec{v}_1 + a_2 \vec{v}_2 + \dots + a_n \vec{v}_n$, with $a_1, a_2, \dots, a_n \in F$

$$[\vec{v}]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in F^n$$

• Matrix representation of a $T: V \to W$, $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$, $\gamma = \{\vec{w}_1, \dots, \vec{w}_m\}$

$$[T]_\beta^\gamma = \overbrace{\begin{pmatrix} | & & | \\ [T(\vec{v}_1)]_\gamma & \cdots & [T(\vec{v}_n)]_\gamma \\ | & & | \end{pmatrix}}^{n} \Big\}\, m \in M_{m \times n}$$

Example: $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by:

$$T(A) \overset{\text{def}}{:=} A^T + 2A$$

tranpose

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \text{ — ordered basis}$$

$$T(\beta) = \left\{ \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \right\}$$

$$[T]_\beta = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

<u>Example:</u> $T: P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by:

$$T(f) \stackrel{def}{=} \begin{pmatrix} f(0) & f(1) \\ 0 & f'(0) \end{pmatrix}$$

Consider ordered basis: $\beta = \{1, x, x^2\}$ for $P_2(\mathbb{R})$

$$\gamma = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

$$T(\beta) = \left\{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}$$

$$\therefore [T]_\beta^\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in M_{4\times 3}$$

Composition of linear transformations and matrix multiplication

Thm: Let V and W be two vector spaces over the same field F.
And let $T: V \to W$ and $U: W \to Z$ be linear.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[rr, bend left, "UT"] & W \arrow[r, "U"] & Z \end{tikzcd}$$

α n β m γ p

(i) Then the composition $UT: V \to Z$ is linear.

(ii) If V, W, Z have ordered bases α, β, γ respectively,

then: $$\underset{M_{p\times n}}{[UT]_\alpha^\gamma} = \underbrace{\underset{M_{p\times m}}{[U]_\beta^\gamma}\,[T]_\alpha^\beta}_{\text{matrix multiplication}} \in M_{m\times n}$$

(i) Let $\vec{x}, \vec{y} \in V$ and $a \in F$. Then:

$$UT(a\vec{x} + \vec{y}) = U\big(aT(\vec{x}) + T(\vec{y})\big) = aUT(x) + UT(\vec{y})$$

$\therefore$ UT is linear.

(ii) Suppose

$$\alpha = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$$
$$\beta = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m\}$$
$$\gamma = \{\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_p\}$$

$$[U]_\beta^\gamma = \underset{\substack{\uparrow \\ M_{p\times m}(F)}}{A} \overset{\text{def}}{=} (a_{ik})_{\substack{1\le i\le p \\ 1\le k\le m}} \quad \text{means:} \quad U(\vec{w}_k) = \sum_{i=1}^{p} a_{ik}\vec{z}_i$$

$$1 \le k \le m$$

$$= \begin{array}{c} \begin{pmatrix} & a_{1k} & \\ & a_{2k} & \\ & \vdots & \\ & a_{pk} & \end{pmatrix} \\ \uparrow \\ k\text{th} \end{array}$$

$$[T]_\alpha^\beta = \underset{\underset{M_{m\times n}(F)}{\in}}{B} \overset{\text{def}}{=} (b_{kj})_{\substack{1\le k\le m\\ 1\le j\le n}} \quad \text{means} \quad T(\vec{v}_j) = \sum_{k=1}^{m} b_{kj}\,\vec{w}_k$$

for $1 \le j \le n$

Then:
$$UT(\vec{v}_j) = U\left(\sum_{k=1}^{m} b_{kj}\,\vec{w}_k\right)$$
$$= \sum_{k=1}^{m} b_{kj}\, U(\vec{w}_k)$$
$$= \sum_{k=1}^{m} b_{kj}\left(\sum_{i=1}^{p} a_{ik}\,\vec{z}_i\right) = \sum_{i=1}^{p} \underbrace{\boxed{\left(\sum_{k=1}^{m} a_{ik} b_{kj}\right)}}_{(i,j)\text{-entry of } AB} \vec{z}_i$$

So, $[UT]_\alpha^\gamma = AB = [U]_\beta^\gamma\,[T]_\alpha^\beta$

Example: Consider: $\mathcal{U} : \overset{\alpha}{P_n}(\mathbb{R}) \to \overset{\beta}{P_{n-1}}(\mathbb{R})$ defined by:

$$\mathcal{U}(f(x)) \overset{def}{=} f'(x)$$

Consider $T : \overset{\beta}{P_{n-1}}(\mathbb{R}) \to \overset{\alpha}{P_n}(\mathbb{R})$ defined by:

$$T(f(x)) \overset{def}{=} \int_0^x f(t)\,dt$$

Let α and β be standard ordered bases for $P_n(\mathbb{R})$ and $P_{n-1}(\mathbb{R})$ respectively.

$$[U]_\alpha^\beta = \begin{pmatrix} 0 & 1 & 0 & 0 & & 0 \\ \vdots & 0 & 2 & 0 & & 0 \\ \vdots & \vdots & 0 & 3 & & \vdots \\ \vdots & \vdots & \vdots & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & 0 \\ 0 & 0 & 0 & 0 & & n \end{pmatrix} \text{ and } [T]_\beta^\alpha = \begin{pmatrix} 0 & 0 & 0 & & 0 \\ 1 & 0 & 0 & & 0 \\ 0 & \frac{1}{2} & 0 & & \vdots \\ \vdots & 0 & \frac{1}{3} & \cdots & \vdots \\ \vdots & \vdots & 0 & & 0 \\ 0 & 0 & \vdots & & \frac{1}{n} \\ & & 0 & & \end{pmatrix}$$

So, $[UT]_\beta = [U]_\alpha^\beta [T]_\beta^\alpha = \underset{\text{matrix}}{\text{Identity}} = [I_{P_{n-1}(\mathbb{R})}]_\beta$

<u>Corollary:</u> Let V and W be finite-dimensional vector spaces with ordered basis β and γ respectively. Let $T: V \to W$ be linear. Then: for any $\vec{u} \in V$, we have:

$$\underbrace{[\,T(\underset{W}{\vec{u}})\,]_\gamma}_{\text{Lin. Transf.}} = \underbrace{[\,T\,]_\beta^\gamma \, [\vec{u}]_\beta}_{\text{Matrix multiplication}}$$

<u>Proof:</u> Fix $\vec{u} \in V$ and consider two linear transformations:

$f: \overset{\alpha}{F} \to \overset{\beta}{V}$ defined by $f(\underset{F}{a}) = a\vec{u} \in V$

$g: F \to \overset{\gamma}{W}$ defined by $g(a) = a\,T(\vec{u}) \in W$

f and g are linear transformations. Also, $g = T \circ f$.

Let $\alpha = \{1\}$ be the standard ordered basis for F.

$$[T(\vec{u})]_\gamma = [g(1)]_\gamma = [\underbrace{g}_{\substack{\shortparallel \\ T\circ f}}]^\gamma_\alpha = [T]^\gamma_\beta [f]^\beta_\alpha = [T]^\gamma_\beta [\overset{\shortparallel}{f(1)}]_\beta$$

$$= [T]^\gamma_\beta [\overset{\shortparallel}{\vec{u}}]_\beta$$

$$\begin{tikzcd} \overset{\alpha}{F} \arrow[r, "f"] \arrow[rr, bend right, "g = T\circ f"'] & \overset{\beta}{V} \arrow[r, "T"] & \overset{\gamma}{W} \end{tikzcd}$$

$$\begin{tikzcd} \overset{\alpha = \{1\}}{F} \arrow[r, "g"] & \overset{\gamma}{W} \end{tikzcd}$$

$$[g]^\gamma_\alpha = \cancel{\begin{pmatrix} | \\ [g(\vec{v}_j)]_\gamma \\ | \end{pmatrix}} = \begin{pmatrix} | \\ [g(1)]_\gamma \\ | \end{pmatrix}$$

Example: $T: M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ defined by:

$$T(A) \overset{\text{def}}{=} A^T + 2A.$$

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$[T]_\beta = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Let $B = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$.

$$\Big|[T(B)]_\beta\Big| = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \Big|[B]_\beta\Big| = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ 4 \\ 0 \end{pmatrix}$$

$$T(B) = \begin{pmatrix} 9 & 5 \\ 4 & 0 \end{pmatrix}$$