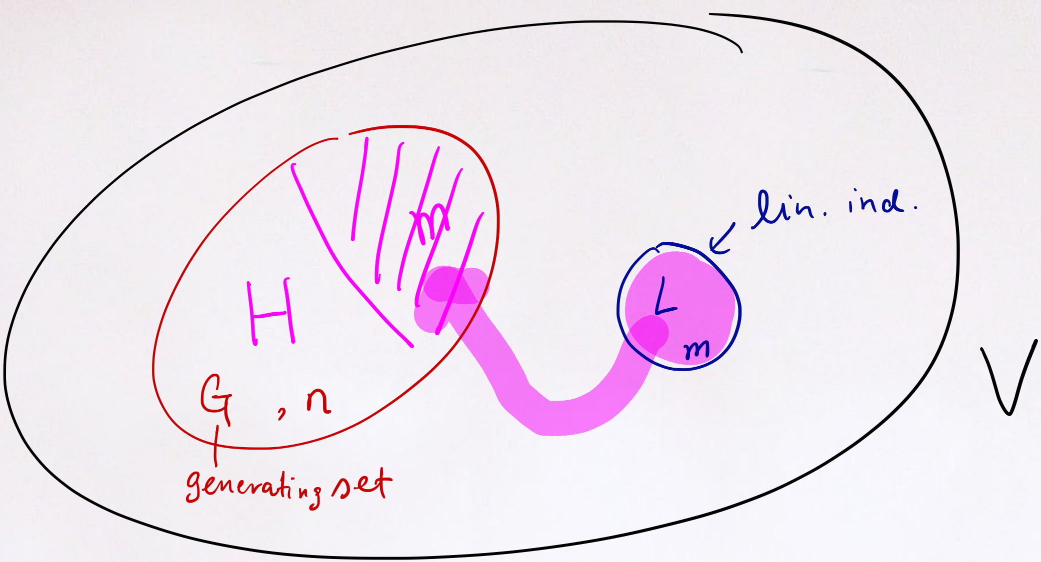
Lecture 6:

Recall:



① $m \le n$

② $L \cup \underset{\overset{\cap}{G}}{H}$ span V.

Dimension

Cor 1: Let V be a vector space having a finite basis. Then, every basis of V contains the same number of vectors.

Pf: Let β and γ be two bases of V.

Since β spans V and γ is lin. independent,

then $|\gamma| \leq |\beta|$ (by replacement Thm)

Similarly, $|\beta| \leq |\gamma|$

$\Rightarrow |\gamma| = |\beta|$.

Definition: A vector space V is called *finite-dimensional* if it has a finite basis. The *dimension* of V, denoted as $\dim(V)$, is the number of vectors in a basis for V.

A vector space which is not finite-dimensional is called *infinite-dimensional*.

<u>Example:</u>

- F^n has dimension n.
- $P_n(F)$ has dimension $n+1$

  $P(F)$ is infinite dimensional
- $M_{n\times n}(F)$ has dimension n^2
- $\mathbb{C}$ is a vector space over $F=\mathbb{C}$, has dimension $\underline{1}$

  (Basis $\beta = \{1\}$)

  $\mathbb{C}$ is a vector space over $F = \mathbb{R}$ has dimension 2

  $(\beta = \{1, i\})$

Cor: Let V be a n-dimensional vector space.

(a) Any finite spanning set has at least n vectors, and a spanning set with exactly n vectors is a basis for V.

(b) Any linearly independent subset of V of size n is a basis for V.

(c) Every linearly independent subset of V can be extended to a basis of V.

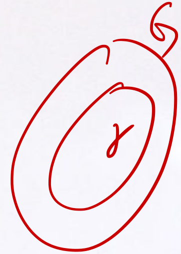
Pf: Let β be a basis for V $(|\beta| = n)$

(a) Let G be a finite spanning set for V. Then:

$\exists\, \gamma \subset G$ which is a basis for V (last time)

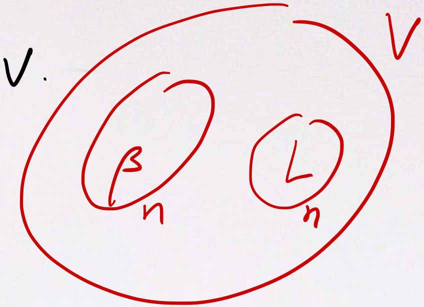
$\therefore\ |\gamma| = n \leq |G|$

If $|G| = |\gamma|$, then: $G = \gamma$ is a basis for V.



(b) Let $L \subset V$ be a lin. independent subset of size n.

By the replacement theorem, $\exists H \subset \beta$ of size $\underbrace{|\beta|}_{=n} - \underbrace{|L|}_{=n} = 0$ such that $L \cup H$ span V.



But $|H| = 0$. $\therefore H = \phi$.

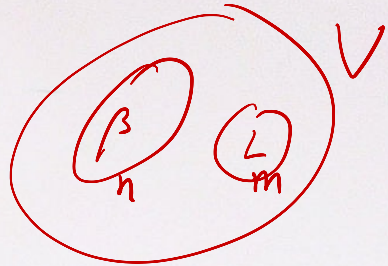
$\therefore L \cup H = L \cup \phi = L$ span V.

$\therefore L$ is a basis

(c) Let $L \subset V$ be a linearly independent subset of size m.

By the replacement theorem, ① $m \le n$

② $\exists H \subset \beta$ of size $n-m$ such that $\underbrace{L}_{m} \cup \underbrace{H}_{n-m}$ span V.



$\therefore |L \cup H| = m + n - m = n$

By (a), $L \cup H$ is a spanning set with exactly $\underset{\dim(V)}{\underset{\shortparallel}{n}}$ elements $\Rightarrow$ $L \cup H$ is a basis.

Theorem: Let V be a finite-dimensional vector space.
Then any subspace $W \subset V$ is finite-dimensional and $\dim(W) \le \dim(V)$
Moreover, if $\dim(W) = \dim(V)$, then $W = V$.

Proof: Let $n = \dim(V)$. If $W = \{\vec{0}\}$, then

$$\dim(W) = 0 \le n \quad (\operatorname{span}(\phi) = \{\vec{0}\})$$

Otherwise, W contains a non-zero $\vec{u}_1$. So, $\{\vec{u}_1\}$ is lin. indep subset of W. If $\operatorname{span}\{\vec{u}_1\} = W$. then $\{\vec{u}_1\}$ is a basis of W.
Otherwise, choose $\vec{u}_2 \in W \setminus \operatorname{Span}\{\vec{u}_1\}$. Then, by a thm before,
$\{\vec{u}_1, \vec{u}_2\}$ is linearly indep. of W.
If $\{\vec{u}_1, \vec{u}_2\}$ span W, then $\{\vec{u}_1, \vec{u}_2\}$ is a basis.

Repeat this process to produce a larger and larger lin. indep.

But a lin. indep. subset of V has size $\leq n$.

The above process must stop, say, it stops at

producing a lin. indep. subset $\beta = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k\}$ of W

with $k \leq n$. Then, β is a basis of W.

$\therefore \dim(W) = k \leq n$.

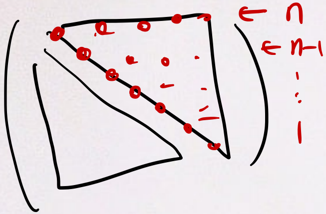
If $\dim(W) = \dim(V) = n$ and $\beta = \{\overset{\in W \subset V}{\vec{u}_1}, \overset{\in W \subset V}{\vec{u}_2}, \dots, \overset{\in W \subset V}{\vec{u}_n}\}$

is a lin. indep. subset of V. Then, β is a basis of V

and $W = \operatorname{span}(\beta) = V$ //

e.g. Set W of symmetric $n \times n$ matrices $(A^t = A)$ is a subspace of $M_{n\times n}(F)$ of dim $= 1 + 2 + \cdots + (n-1) + n$

$$= \frac{1}{2}(n)(n+1) \leq \underset{\dim(V)}{\underset{\shortparallel}{n^2}}$$



<u>Corollary</u>: If W is a subspace of a fin-dimensional vector space V, then any basis of W can be extended to a basis of V.

<u>Proof</u>: Exercise

Linear Transformation

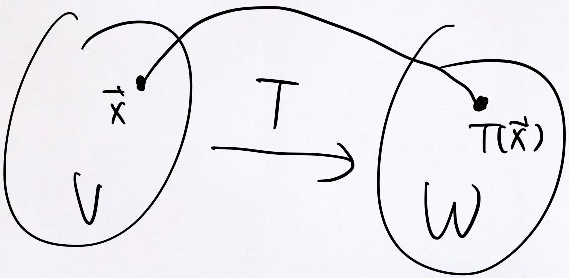
Definition: Let V and W be vector spaces over F.

A linear transformation from V to W is a map $T: V \to W$

such that: (a) $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$

(b) $T(a\vec{x}) = aT(\vec{x})$

for all $\vec{x}, \vec{y} \in V$, $a \in F$.



Proposition: Let $T: V \to W$ be a linear transformation. Then:

(i) $T(\vec{0}_V) = \vec{0}_W$

(ii) $T\left(\sum_{i=1}^{n} a_i \vec{x}_i\right) = \sum_{i=1}^{n} a_i T(\vec{x}_i) \quad \forall\, \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n \in V$

$a_1, a_2, \ldots, a_n \in F$.

(T preserves linear combination)

(i) $T(\vec{0}_V) = T(\vec{0}_V + \vec{0}_V) = T(\vec{0}_V) + T(\vec{0}_V)$

$\Rightarrow T(\vec{0}_V) = \vec{0}_W$. (Cancellation law)

(ii) Use math. induction (exercise)

Examples: • For any vector spaces V and W, we have:

(a) The zero transformation $T_0 : V \to W$ defined by $T_0(\vec{x}) := \vec{0}_W$ for $\forall \vec{x} \in V$

(b) The identity transformation $I_V : V \to V$ defined by $I_V(\vec{x}) = \vec{x}$ for $\forall \vec{x} \in V$.

• Let $A \in M_{m\times n}(F)$ be a $m\times n$ matrice in F.

Define: $L_A : F^n \to F^m$ as: (F^n = space of col vectors of size n)

$$L_A(\vec{x}) \overset{def}{=} A\vec{x}$$

L_A is called the left multiplication by A.

• $T : M_{m\times n}(F) \to M_{n\times m}(F)$ defined by $T(A) \overset{def}{=} A^t$ (transpose of A)

- $T: P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ defined by $T(f(x)) = f'(x)$ (derivative of f) is a lin. transf.

- Let a and $b \in \mathbb{R}$, $a < b$. Then,

  $T: C(\mathbb{R}) \to \mathbb{R}$ defined by:
  (Space of continuous functions)

$$T(f) \overset{def}{=} \int_a^b f(t)\,dt$$

Null space or Range

Definition: Let V and W be vector spaces and $T: V \to W$ be a linear transformation.

Then, the null space (or kernel) of T is defined as:

$$N(T) :\overset{\text{def}}{=} \{ \vec{x} \in V : T(\vec{x}) = \vec{0}_W \} \subset V$$

the range (or image) of T is defined as:

$$R(T) := \{ T(\vec{x}) : \vec{x} \in V \} \subset W$$

e.g. For $I_V : V \to V$, $N(I_V) = \{ \vec{0}_V \}$, $R(I_V) = V$
(identity)

For $T_0 : V \to W$, $N(T_0) = V$, $R(T_0) = \{ \vec{0}_W \}$
(zero transf)

- $L_A : F^n \to F^m \quad (A \in M_{m\times n}(F))$

  $N(L_A) = N(A) =$ null space of A

  $R(L_A) = \mathcal{C}(A) =$ col space of A (space of linear combination of col vectors of A)

  $$A = \begin{pmatrix} | & | & \cdots & | \end{pmatrix}$$

- For $T : P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ defined by $T(f(x)) = f'(x)$, then:

  $N(T) = \{ a_0 \in P_n(\mathbb{R}) : a_0 \in \mathbb{R} \}$

  $R(T) = P_{n-1}(\mathbb{R})$

Proposition: Let $T: V \to W$ be a linear transformation.

Then: $N(T)$ and $R(T)$ are subspaces of V and W respectively.

Proof: $\because$ $T(\vec{0}_V) = \vec{0}_W$

$\therefore$ $\vec{0}_V \in N(T)$ and $\vec{0}_W \in R(T)$

Let $\vec{x}$ and $\vec{y} \in N(T)$ and $a \in F$. Then:

$$T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y}) = \vec{0}_W + \vec{0}_W = \vec{0}_W \quad \text{and}$$

$$T(a\vec{x}) = aT(\vec{x}) = a\vec{0}_W = \vec{0}_W$$

$\therefore$ $\vec{x}+\vec{y} \in N(T)$ and $a\vec{x} \in N(T)$

$\therefore$ $N(T)$ is a subspace of V.

Now, let $\vec{u}, \vec{v} \in R(T)$ and $a \in F$.

Then: $\exists\, \vec{x}, \vec{y} \in V$ such that $\vec{u} = T(\vec{x})$ and $\vec{v} = T(\vec{y})$

So, $T(\underbrace{\vec{x}+\vec{y}}_{}{}^{\in V}) = T(\vec{x}) + T(\vec{y}) = \vec{u} + \vec{v} \Rightarrow \vec{u} + \vec{v} \in R(T)$

$T(\overset{\in V}{a\vec{x}}) = aT(\vec{x}) = a\vec{u} \Rightarrow a\vec{u} \in R(T)$

$\therefore$ $R(T)$ is a subspace of W.

Remark: $T: V \to W$ is onto iff $R(T) = W$

(follows from the def)

Proposition: A linear transformation $T: V \to W$ is one-to-one iff $N(T) = \{\vec{0}\}$.

Pf: (Recap: One-to-one $\Longleftrightarrow$ " $T(\vec{x}) = T(\vec{y}) \Rightarrow \vec{x} = \vec{y}$ "

$(\Rightarrow)$ If T is one-to-one, then: for any $\vec{x} \in N(T)$, we have

$$T(\vec{x}) = \vec{0}_W = T(\vec{0}_V)$$

$$\Rightarrow \quad \vec{x} = \vec{0}_V$$

This implies $N(T) = \{\vec{0}_V\}$.

($\Leftarrow$) Suppose $N(T) = \{\vec{0}_V\}$

Let $\vec{x}, \vec{y} \in V$ such that $T(\vec{x}) = T(\vec{y})$.

Then: $T(\vec{x}) - T(\vec{y}) = T(\vec{x} - \vec{y}) = \vec{0}$

This implies $\vec{x} - \vec{y} \in N(T) = \{\vec{0}_V\}$

$\therefore \vec{x} - \vec{y} = \vec{0}_V$ or $\vec{x} = \vec{y}$.

$\therefore$ T is 1-1.