Lecture 23:

Def: Let T be a linear operator on an inner product space V. We say T is self-adjoint (Hermitian) if $T^* = T$.
An $n \times n$ real or complex matrix A is called self-adjoint (or Hermitian) if $A^* = A$.

Lemma: Let T be a self-adjoint linear operator on a fin-dim inner product space V. Then:

(a) Every eigenvalue of T is real.

(b) Suppose V is real inner product space. Then, the char. poly of T splits over $\mathbb{R}$.

Proof: (a) Suppose $T(\vec{x}) = \lambda \vec{x}$ for $\vec{x} \neq \vec{0}$.

Then: $T^*(\vec{x}) = \bar{\lambda}\vec{x}$ ($\because$ T is normal)

$\therefore \lambda \vec{x} = T(\vec{x}) = T^*(\vec{x}) = \bar{\lambda}\vec{x}$

$\therefore (\lambda - \bar{\lambda}) \underset{\neq \vec{0}}{\vec{x}} = \vec{0} \Rightarrow \lambda = \bar{\lambda}$. $\therefore \lambda$ is real.

(b) Let $n = \dim(V)$, β be an orthonormal basis for V and let $A \overset{\text{def}}{=} [T]_\beta$

Then: A is self-adjoint. Consider: $L_A : \mathbb{C}^n \to \mathbb{C}^n$

By (a), the eigenvalues of L_A are real.

By Fundamental Thm of Algebra, $f_{L_A}(t)$ splits into factors of the form $t - \lambda$ where λ is an eigenvalue of L_A.

$\because$ λ is real $\therefore$ $f_{L_A}(t)$ splits over $\mathbb{R}$.

But $f_T(t) = f_{L_A}(t)$. So, the result follows.

Theorem: Let T be a linear operator on a fin-dim real inner product space V. Then T is self-adjoint iff $\exists$ orthonormal basis for V consisting of eigenvectors of T.

Proof: ($\Rightarrow$) Suppose T is self-adjoint. By the Lemma, the char poly of T splits over $\mathbb{R}$. By Schur's Theorem, $\exists$ an orthonormal basis β for V s.t. $A \overset{\text{def}}{=} [T]_\beta$ is upper triangular. But:

$$A^* = ([T]_\beta)^* = [T^*]_\beta = [T]_\beta = A$$

So, A is both upper triangular and lower triangular.

Hence, A is diagonal.

$\therefore$ β consists of eigenvectors of T.

($\Leftarrow$) Suppose $\exists$ orthurnormal basis β for V s.t. $A = [T]_\beta$ is diagonal.

Then: $[T^*]_\beta = ([T]_\beta)^* = A^t = A = [T]_\beta$

$\therefore \quad T^* = T$

$\therefore \quad T$ is self-adjoint.

e.g.

Consider $A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$. Then $\exists P \in O(3)$ s.t. $P^t A P$ is diagonal.

To find P explicitly, we first compute the eigenvalues of A:

$$f_A(t) = (8-t)(2-t)^2$$

So the eigenvalues are $\lambda = 2$ and $\lambda = 8$

For $\lambda = 8$, $(1,1,1)$ is an eigenvector

For $\lambda = 2$, $\{(-1,1,0), (-1,0,1)\}$ is a basis for the eigenspace E_2 but it is not orthogonal.

Applying the Gram-Schmidt process produces the orthogonal basis $\{(-1,1,0), (1,1,-2)\}$ of E_2.

Then an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of A is given by

$$\left\{ \frac{1}{\sqrt{2}}(-1,1,0), \frac{1}{\sqrt{6}}(1,1,-2), \frac{1}{\sqrt{3}}(1,1,1) \right\}$$

which gives P as

$$P = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2\sqrt{6} & 1/\sqrt{3} \end{pmatrix}.$$

Def: Let T be a linear operator on finite-dim inner product space V over F. If $\|T(\vec{x})\| = \|\vec{x}\|$ $\forall \vec{x} \in V$, then we call T is a unitary linear operator. (resp. orthogonal operator) if $F = \mathbb{C}$ (resp $F = \mathbb{R}$)

Lemma: Let U be a self-adjoint linear operator on a fin-dim inner product space V. If $\langle \vec{x}, U(\vec{x}) \rangle = 0$ $\forall \vec{x} \in V$, then $U = T_0$ = zero transf.

Pf: Choose an orthonormal basis β for V consisting of eigenvectors of U.

If $\vec{x} \in \beta$, then $U(\vec{x}) = \lambda \vec{x}$ for some λ.

$$0 = \langle \vec{x}, U(\vec{x}) \rangle = \langle \vec{x}, \lambda \vec{x} \rangle = \bar{\lambda} \langle \vec{x}, \vec{x} \rangle = \bar{\lambda} \|\vec{x}\|^2$$

$$\Rightarrow \quad \lambda = 0$$

$\therefore \quad U(\vec{x}) = 0$ for $\forall \vec{x} \in \beta$

$\therefore \quad U = T_0$.

Thm: For a linear operator T on a fin-dim inner product space V, the following are equivalent:

(a) $TT^* = T^*T = I$

(b) T preserves the inner product on V, i.e.,

$$\langle T(\vec{x}), T(\vec{y}) \rangle = \langle \vec{x}, \vec{y} \rangle \quad \forall \vec{x}, \vec{y} \in V.$$

(c) $T(\underbrace{\beta}_{=\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}}) \stackrel{\text{def}}{=} \{T(\vec{v}_1), \dots, T(\vec{v}_n)\}$ is an orthonormal basis for V for any orthonormal basis β for V

(d) $\exists$ an orthonormal basis β for V s.t. $T(\beta)$ is an orthonormal basis for V.

(e) $\|T(\vec{x})\| = \|\vec{x}\|$ for $\forall \vec{x} \in V$