

Lecture 22:

Recap: If Char poly of T splits,

then: \exists o.n. basis $\beta \Rightarrow [T]_{\beta}$ is upper triangular

Goal: Find orthonormal basis β consisting of eigenvectors!!

Observation:

Assume T is diagonalizable and assume \exists an orthonormal basis β for V s.t. $[T]_{\beta}$ is diagonal.

Then: $[T^*]_{\beta} = ([T]_{\beta})^*$ is also diagonal

$$\therefore ([T]_{\beta})^* ([T]_{\beta}) = ([T]_{\beta}) ([T]_{\beta})^*$$

$$\overset{''}{[T^*]_{\beta}} [T]_{\beta} = [T]_{\beta} [T^*]_{\beta}$$

$$\overset{''}{[T^* T]_{\beta}} = [T T^*]_{\beta}$$

$$\Rightarrow T^* T = T T^*$$

Definition: Let V be an inner product space. We say that a linear operator $T: V \rightarrow V$ is **normal** if $T^*T = TT^*$.

An $n \times n$ real or complex matrix A is called **normal** if

$$A^*A = AA^*$$

Example:

- Unitary (when $F = \mathbb{C}$) or orthogonal (when $F = \mathbb{R}$) if $T^*T = TT^* = I$
- Hermitian (or self-adjoint) if $T^* = T$
- Skew-Hermitian (or anti-self-adjoint) if $T^* = -T$.

Are normal!

e.g. The rotation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by θ , can be represented

by $A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ is orthogonal ($A^*A = AA^* = I$)

So, it is normal.

Proposition: Let V be an inner product space, and let T be a normal linear operator on V . Then: we have:

(a) $\|T(\vec{x})\| = \|T^*(\vec{x})\| \quad \forall x \in V$

(b) $T - cI$ is normal $\forall c \in F$.

(c) If $T(\vec{x}) = \lambda\vec{x}$, then: $T^*(\vec{x}) = \overline{\lambda}\vec{x}$

(d) If λ_1 and λ_2 are distinct eigenvalues of T with corresponding eigenvectors \vec{x}_1 and \vec{x}_2 , then:

\vec{x}_1 and \vec{x}_2 are orthogonal.

Proof: (a) $\forall \vec{x} \in V$, we have:

$$\begin{aligned}\|T(\vec{x})\|^2 &= \langle T(\vec{x}), T(\vec{x}) \rangle = \langle T^* T(\vec{x}), \vec{x} \rangle \\ &= \langle T T^*(\vec{x}), \vec{x} \rangle = \langle T^*(\vec{x}), T^*(\vec{x}) \rangle \\ &= \|T^*(\vec{x})\|^2\end{aligned}$$

$$\begin{aligned}\text{(b). } (T - cI)(T - cI)^* &= (T - cI)(T^* - \bar{c}I) \\ &= TT^* - cT^* - \bar{c}T + c\bar{c}I \\ &= T^*T - cT^* - \bar{c}T + c\bar{c}I \\ &= (T - cI)^*(T - cI).\end{aligned}$$

(c) Suppose $T(\vec{x}) = \lambda \vec{x}$. Let $U = T - \lambda I$. Then, U is normal (by (b)) and $U(\vec{x}) = \vec{0}$. So, by (a),

$$0 = \|U(\vec{x})\| = \|U^*(\vec{x})\| = \|(T^* - \bar{\lambda}I)(\vec{x})\| \Leftrightarrow T^*(\vec{x}) = \bar{\lambda} \vec{x}.$$

(d) By (c), we have:

$$\begin{aligned}\lambda_1 \langle \vec{x}_1, \vec{x}_2 \rangle &= \langle T(\vec{x}_1), \vec{x}_2 \rangle = \langle \vec{x}_1, T^*(\vec{x}_2) \rangle \\ &= \langle \vec{x}_1, \lambda_2 \vec{x}_2 \rangle \\ &= \lambda_2 \langle \vec{x}_1, \vec{x}_2 \rangle\end{aligned}$$

$\lambda_1 \neq \lambda_2$

$$\Leftrightarrow (\lambda_1 - \lambda_2) \langle \vec{x}_1, \vec{x}_2 \rangle = 0$$

$$\Rightarrow \langle \vec{x}_1, \vec{x}_2 \rangle = 0$$

Theorem: Let T be a linear operator on a finite-dim complex inner product space V . Then, T is normal iff \exists an orthonormal basis for V consisting of eigenvectors of T .

Proof: (\Leftarrow) Obvious.

(\Rightarrow) Suppose T is normal.

By the Fundamental Thm of algebra, $f_T(t)$ splits.

\therefore Schur's Theorem gives us an orthonormal basis

$\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ s.t. $[T]_\beta$ is upper triangular.

$[T]_\beta = \left(\begin{array}{c|c} [T]_{\beta_1} & \\ \hline & \end{array} \right)$. In particular, \vec{v}_1 is an eigenvector of T .

Suppose that $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{k-1}$ are eigenvectors of T and $\lambda_1, \lambda_2, \dots, \lambda_{k-1}$ are their corresponding eigenvalues

We claim that \vec{v}_k is an eigenvector of T (so by induction, all vectors in β are eigenvectors of T)

$$\text{Now, } T(\vec{v}_j) = \lambda_j \vec{v}_j \Rightarrow T^*(\vec{v}_j) = \bar{\lambda}_j \vec{v}_j \text{ for } j=1, 2, \dots, k-1$$

$\therefore A \stackrel{\text{def}}{=} [T]_\beta$ is upper triangular

$$T(\vec{v}_k) = A_{1k} \vec{v}_1 + A_{2k} \vec{v}_2 + \dots + A_{kk} \vec{v}_k$$

$$\text{But : } A_{jk} = \langle T(\vec{v}_k), \vec{v}_j \rangle = \langle \vec{v}_k, T^*(\vec{v}_j) \rangle = \langle \vec{v}_k, \bar{\lambda}_j \vec{v}_j \rangle \\ = \lambda_j \langle \vec{v}_k, \vec{v}_j \rangle \\ = 0$$

for $j=1, 2, \dots, k-1$. $\therefore T(\vec{v}_k) = A_{kk} \vec{v}_k$
 $\therefore \vec{v}_k = \text{eigenvector of } T.$

Example: Let H be the set of continuous complex-valued functions defined on $[0, 2\pi]$ equipped w/ the inner product

$$\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(t) \overline{g(t)} dt \quad \text{for } f, g \in H.$$

and the orthonormal subset:

$$S = \{ f_n(t) := e^{int} = n \in \mathbb{Z} \} \subset H$$

"
($\cos nt + i \sin nt$)

inf dim



Let $V = \text{span}(S)$ and consider the operators T and U on V

defined by: $T(f) = f_1 \cdot f$, $U(f) = f_{-1} f$

$= e^{it} f$ $e^{-it} f$

$$\therefore T(f_n) = f_{n+1} \quad \text{and} \quad U(f_n) = f_{n-1} \quad \forall n \in \mathbb{Z}.$$

e^{it} "int
 $e^{i(n+1)t}$

$\begin{cases} 1 & \text{if } m+1=n \\ 0 & \text{otherwise} \end{cases}$

Then: $\langle T(f_m), f_n \rangle = \langle f_{m+1}, f_n \rangle = \delta_{m+1, n}$

$= \delta_{m, n-1}$

$= \langle f_m, f_{n-1} \rangle$

$= \langle f_m, U(f_n) \rangle$

$\Rightarrow U = T^*$

$\therefore TT^* = TU = I = T^*T$ $\therefore T$ is normal.

However, T has no eigenvectors.

If $f \in V$ is an eigenvector of T , say, $T(f) = \lambda f$ ($\lambda \in \mathbb{C}$)

Then, we write $f = \sum_{i=n}^m a_i f_i$, where $a_m \neq 0$

$$\therefore \sum_{i=n}^m a_i f_{i+1} = T(f) = \lambda f = \sum_{i=n}^m \lambda a_i f_i$$

$$\Rightarrow f_{m+1} = \frac{1}{a_m} \left(\lambda a_m f_n + \sum_{i=n+1}^m (\lambda a_i - a_{i-1}) f_i \right)$$

Contradicting the fact that S is linearly independent.

Def: Let T be a linear operator on an inner product space V . We say T is self-adjoint (Hermitian) if $T^* = T$.
An $n \times n$ real or complex matrix A is called self-adjoint (or Hermitian) if $A^* = A$.

Next time:

Thm: Let T be a lin. operator on a fin-dim **real** inner product space V . Then: T is self-adjoint iff \exists orthonormal basis for V consisting of eigenvectors of T .

Next time: If $T: V \rightarrow V$ ^{fin-dim} is self-adjoint, then:
Every eigenvalues of T are real.