MATH2040A/B Homework 9 Solution

(Sec 6.1 Q9) Ans :

if $\langle x, z\rangle = 0$ for all $z \in \beta$, writing $x = \sum_{i=1}^{m} x_i z_i$, then $\langle x, x\rangle = 0 \implies \|x\| = 0 \implies x = 0$.

ii. $\langle x, z\rangle = \langle y, z\rangle$ iff $\langle x - y, z\rangle = 0$, by above, we have $x - y = 0$.

(Sec 6.1 Q10 and Q12) Ans: We prove only Q12 since Q10 is a special case of Q12. We prove by induction. For $k = 1$ it is trivial. Suppose $k = n$ is true, then we have

$$\begin{aligned}\|\sum_{i=1}^{n+1} a_i v_i\|^2 &= \|\sum_{i=1}^{n} a_i v_i + a_{n+1}v_{n+1}\|^2 \\ &= \|\sum_{i=1}^{n} a_i v_i\|^2 + 2Re\langle\sum_{i=1}^{n} a_i v_i, a_{n+1}v_{n+1}\rangle + \|a_{n+1}v_{n+1}\|^2 \\ &= \sum_{i=1}^{n+1} |a_i|^2\|v_i\|^2.\end{aligned}$$

(Sec 6.1 Q11) Ans :

$$\begin{aligned}\|x+y\|^2 + \|x-y\|^2 &= \|x\|^2 + 2Re\langle x, y\rangle + \|y\|^2 + \|x\|^2 - 2Re\langle x, y\rangle + \|y\|^2 \\ &= 2\|x\|^2 + 2\|y\|^2.\end{aligned}$$

(Sec 6.1 Q15(a)) **If:** Without loss the generality, we assume $x = ay$. Then

$$|\langle x, y\rangle| = |\langle ay, y\rangle| = |a||\langle y, y\rangle| = |a|||y||^2 = ||x|| \cdot ||y||$$

Only if: If the identity holds and $y \neq 0$ (if $y = 0, y = 0 \cdot x$. The proof is done), we let $a = \frac{\langle x,y\rangle}{||y||^2}$ and $z = x - ay$.

By the equality $|\langle x, y\rangle| = ||x|| \cdot ||y||, |a| = \frac{||x||}{||y||}$. Hence $||x||^2 = |a|^2||y||^2$.

Also, we have $\langle z, y\rangle = \langle x, y\rangle - a\langle y, y\rangle = \langle x, y\rangle - \frac{\langle x,y\rangle}{||y||^2}||y||^2 = 0$, which implies y and z are orthogonal.

By Q10, $||x||^2 = ||ay + z||^2 = ||ay||^2 + ||z||^2 = ||x||^2 + ||z||^2$. Therefore, $z = 0$ which implies $x = ay, a = \frac{\langle x,y\rangle}{||y||^2}$.

(Sec 6.1 Q15(b)) **Claim:** $||x + y|| = ||x|| + ||y||$ if and only if one of the vectors x or y is a non-negative multiple of the other.

Proof:

$$\begin{aligned}||x + y|| = ||x|| + ||y|| &\Leftrightarrow ||x + y||^2 = (||x|| + ||y||)^2 \\ &\Leftrightarrow \langle x, y\rangle = ||x|| \cdot ||y||\end{aligned}$$

Thus we only need to prove $\langle x, y\rangle = ||x|| \cdot ||y||$ if and only if one of the vectors x or y is a non-negative multiple of the other. And the proof of this is similar with the proof in part(a)–only need to replace |a| with a.

Claim: $||x_1 + x_2 \cdots + x_n|| = ||x_1|| + \cdots + ||x_n||$ if and only if there exist one of the vectors $x_k \in \{x_i\}_{i=1}^n$ s.t. for $\forall i = 1, 2, ..., n$, x_i a non-negative multiple of x_k.

Proof:

If: Check it directly.

Only: If all of the vectors equal to 0, we choose $k = 1$ and the proof is done.
We assume $x_k \neq 0$, then

$$\begin{aligned} ||x_1 + x_2 \cdots + x_n|| = ||x_1|| + \cdots + ||x_n| &\Leftrightarrow ||x_1 + x_2 \cdots + x_n||^2 = ||x_1||^2 + \cdots + ||x_n||^2 \\ &\Leftrightarrow \langle x_p, x_q \rangle = ||x_p|| \cdot ||x_q||, \forall 1 \leq p, q \leq n \\ &\Rightarrow \langle x_i, x_k \rangle = ||x_i|| \cdot ||x_k||, \forall 1 \leq i \leq n \end{aligned}$$

Using $x_k \neq 0$ and $\langle x_i, x_k \rangle = ||x_i|| \cdot ||x_k||, \forall 1 \leq i \leq n$, we can prove the desirable result similarly.

(Sec 6.2 Q2(g)) Ans: Let $S' = \{X_1, X_2, X_3\}$ be the obtained orthogonal basis for $span(S)$. By applying GS process we obtain that

$$\begin{aligned} X_1 &= \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix} \\ X_2 &= \begin{pmatrix} -1 & 9 \\ 5 & -1 \end{pmatrix} - 3636 \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix} \\ X_3 &= \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix} - -7236 \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix} - -7272 \begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 9 & -3 \\ 6 & -6 \end{pmatrix}. \end{aligned}$$

Hence the orthonormal basis is

$$\beta = \{16 \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}, 16\sqrt{2} \begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix}, 19\sqrt{2} \begin{pmatrix} 9 & -3 \\ 6 & -6 \end{pmatrix}\}$$

For the matrix A it is just direct checking.

(Sec 6.2 Q2(i)) Ans: We obtain the orthogonal basis $S' = \{X_1, X_2, X_3, X_4\}$ by applying GS process, where

$$\begin{aligned} X_1 &= \sin t \\ X_2 &= \cos t \\ X_3 &= 1 - 2\pi 2 \sin t - 0\pi 2 \cos t = 1 - 4\pi \sin t \\ X_4 &= t - 2 \sin t + 4\pi \cos t - 12(\pi^3 - 8\pi)\pi^2 - 8(1 - 4\pi \sin t) \end{aligned}$$

Hence the orthonomral basis β is

$$\{2\pi \sin t, 2\pi \cos t, 1\pi - 8\pi(1 - 4\pi \sin t), 12\pi\pi^4 - 96\Big[t - 2 \sin t + 4\pi \cos t - 12(\pi^3 - 8\pi)\pi^2 - 8(1 - 4\pi \sin t)\Big]\}$$

The rest is direct calculation.

(Sec 6.2 Q6) By Theorem 6.6, there exists $w \in W$ and $y \in W^\perp$ such that $x = w + y$. Since $x \notin W$, $y \neq \overrightarrow{0}$. Then we have

$$\langle x, y \rangle = \langle w + y, y \rangle = \langle w, y \rangle + \langle y, y \rangle = ||y||^2 > 0.$$

(Sec 6.2 Q8) We prove it by mathematical induction.
When $i = 1$, the conclusion holds obviously.
We assume the conclusion holds when $i = k$, then when $i = k + 1$,

$$\begin{aligned} v_{k+1} &= w_{k+1} - \sum_{j=1}^{k} \frac{\langle w_{k+1}, v_j \rangle}{||v_j||^2} v_j \\ &= w_{k+1} - \sum_{j=1}^{k} \frac{\langle w_{k+1}, w_j \rangle}{||w_j||^2} w_j \quad \text{by induction assumption} \\ &= w_{k+1} \quad \text{by } \langle w_{k+1}, j \rangle = 0, \forall j = 1, ..., k \end{aligned}$$

Thus the conclusion also holds when $i = k + 1$.

(Sec 6.2 Q14) Since $W_1, W_2 \subset W_1 + W_2$, by Q13(a), $(W_1 + W_2)^\perp$ is contained in $W_1^\perp$ and $W_2^\perp$. Therefore $(W_1 + W_2)^\perp \subset W_1^\perp \cap W_2^\perp$.

On the other hand, if $x \in W_1^\perp \cap W_2^\perp$, for all $w \in W_1 + W_2$, there exists $w_1 \in W_1$, $w_2 \in W_2$, such that $w = w_1 + w_2$. Since $\langle x, w_1\rangle = \langle x, w_2\rangle = 0$, we have $\langle x, w\rangle = \langle x, w_1\rangle + \langle x, w_2\rangle = 0$. Therefore $x \in (W_1 + W_2)^\perp$ and hence $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$.

By applying this with W_1 and W_2 replaced by $W_1^\perp$ and $W_2^\perp$ respectively, and applying Q13(c), we have $(W_1^\perp + W_2^\perp)^\perp = (W_1^\perp)^\perp \cap (W_2^\perp)^\perp = W_1 \cap W_2$. By taking orthogonal complement on both sides and applying Q13(c) again, we have $(W_1 \cap W_2)^\perp = ((W_1^\perp + W_2^\perp)^\perp)^\perp = W_1^\perp + W_2^\perp$.

(Sec 6.2 Q17) For all $x \in V$, $T(x) \in V$ and thus $||T(x)||^2 = \langle T(x), T(x)\rangle = 0$ by taking $y = T(x)$. Hence $T(x) = \overrightarrow{0}$ for all $x \in V$ and $T = T_0$ the zero transformation.

Now we suppose $\langle T(x), y\rangle = 0$ for all x and y in some basis β for V. We want to prove that this implies $\langle T(x'), y'\rangle = 0$ for all x' and y' in V.

Since β is a basis, there exists $x_1, \ldots, x_m \in \beta$, $y_1, \ldots, y_n \in \beta$, and scalars $a_1, \ldots, a_m, b_1, \ldots, b_n$ such that

$$x' = \sum_{i=1}^{m} a_i x_i \text{ and } y' = \sum_{j=1}^{n} b_j y_j.$$

Then we have

$$\begin{aligned}
&\langle T(x'), y'\rangle \\
=&\left\langle T\left(\sum_{i=1}^{m} a_i x_i\right), \sum_{j=1}^{n} b_j y_j\right\rangle \\
=&\left\langle \sum_{i=1}^{m} a_i T(x_i), \sum_{j=1}^{n} b_j y_j\right\rangle \\
=&\sum_{i=1}^{m}\sum_{j=1}^{n} a_i \overline{b_j} \langle T(x_i), y_j\rangle \\
=&0.
\end{aligned}$$