Further examples

Def Let $f : [a,b] \to \mathbb{R}$ be a function, and

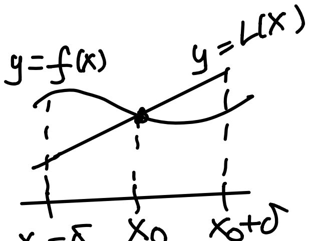
$$L : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \alpha x + \beta \quad \text{for some } \alpha, \beta \in \mathbb{R}.$$

We say <u>L crosses f</u> (or <u>f crosses L</u>)

if $\exists\, x_0 \in [a,b]$ and $\delta > 0$ such that

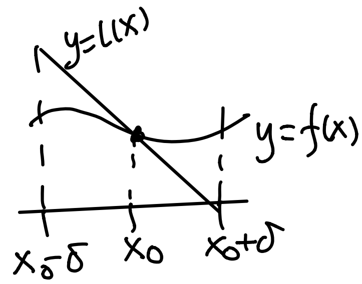
$$f(x_0) = L(x_0)$$

and either one of the following holds:



(i) $L(x) \leq f(x)$, $x \in [x_0 - \delta, x_0] \cap [a,b]$

$L(x) \geq f(x)$, $x \in [x_0, x_0 + \delta] \cap [a,b]$

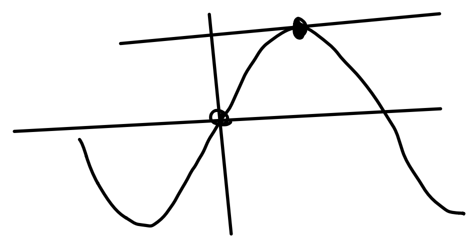


(ii) $L(x) \geq f(x)$, $x \in [x_0 - \delta, x_0] \cap [a,b]$

$L(x) \leq f(x)$, $x \in [x_0, x_0 + \delta] \cap [a,b]$

<u>eg</u>: $f(x) = \sin x$

$L = x$-axis (i.e. $L \equiv 0$)



clearly L crosses f (at many points)

(At $x_0 = 0$, the $\delta > 0$ can be chosen as π)

If $L_1(x) \equiv 1$, L_1 doesn't cross f:

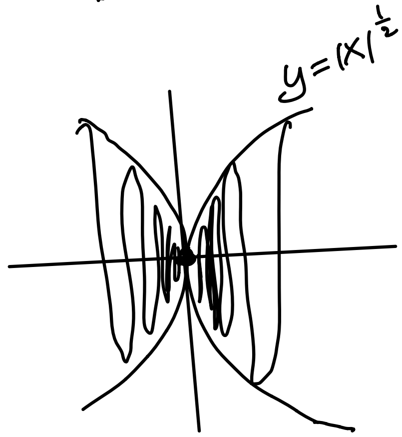
at every intersection $(2n+1)\frac{\pi}{2}$, $f(x) < L_1(x) \equiv 1$

for all x in the nbd $\left((2n+1)\frac{\pi}{2} - \delta, (2n+1)\frac{\pi}{2} + \delta\right)$

except $x = (2n+1)\frac{\pi}{2}$, for any $0 < \delta < 2\pi$.

eg: $f(x) = \begin{cases} |x|^{\frac{1}{2}} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$

Then no "line" L crosses f at $x_0 = 0$.



All "line" L passing thro. $(0,0)$ intersects $y = \pm |x|^{1/2}$. Then infinite oscillation of f $\Rightarrow$ neither (i) nor (ii) in the definition holds.

Def: A function $f: [a,b] \to \mathbb{R}$ is said to be "crosses no lines" if there is no $L(x) = \alpha x + \beta$ crosses f.

Thm The set $Z = \{ f \in C[a,b] : f \text{ crosses no lines} \}$ is a residual set in $C[a,b]$, and hence dense.

Pf: Note that

$$C[a,b] \setminus Z = \{ f \in C[a,b] : \exists \text{ some } L \text{ crosses } f \text{ (at some pt.)} \}$$

where $L(x) = \alpha x + \beta$ $(\alpha, \beta \in \mathbb{R})$.

And we need to show that $C[a,b] \setminus Z$ is of 1st category.

Notation: For $f \in C[a,b]$ and $\alpha \in \mathbb{R}$, we denote

$$f_{-\alpha}(x) = f(x) - \alpha x.$$

(subtracting the linear part of L from f)

Let A_n be the set of $f \in C[a,b]$ for which

$\exists\, \alpha \in [-n, n]$ & $x \in [a,b]$ s.t.

$$\begin{cases} f_{-\alpha}(t) \leq f_{-\alpha}(x), & \forall\, t \in (x - \frac{1}{n}, x) \quad (t \in [a,b]) \\ f_{-\alpha}(t) \geq f_{-\alpha}(x), & \forall\, t \in (x, x + \frac{1}{n}) \quad (t \in [a,b]) \end{cases}$$

Clearly $A_n \subset A_{n+1}$, $\forall n$ $\left(\text{since } (x - \frac{1}{n+1}, x + \frac{1}{n+1}) \subset (x - \frac{1}{n}, x + \frac{1}{n}) \right)$

Note that t is now the independent variable, and

$f_{-\alpha}(t) \leq f_{-\alpha}(x)$ is exactly

$$f(t) - \alpha t \leq f(x) - \alpha x$$

$$\Leftrightarrow \quad f(t) \leq \alpha t + (f(x) - \alpha x) = Lt \qquad \forall\, t \in (x-\tfrac{1}{n}, x)$$

Similarly $\quad f_{-\alpha}(t) \geq f_{-\alpha}(x)$ is exactly

$$f(t) \geq \alpha t + (f(x) - \alpha x) = Lt, \quad \forall\, t \in (x, x+\tfrac{1}{n})$$

$\therefore\ f \in A_n \Rightarrow f$ crosses L at x, $\left(Lt = \alpha t + (f(x) - \alpha x)\right)$

(with $\delta = \frac{1}{n}$ and slope $|\alpha| \leq n$.)

And if f crosses some L at some x, then either

$$f \in A_n \quad \text{or} \quad -f \in A_n \qquad \text{for some } n.$$

$$\Rightarrow \quad f \in A = \bigcup_{n=1}^{\infty} A_n \quad \text{or} \quad -f \in A = \bigcup_{n=1}^{\infty} A_n.$$

Denote $\quad B = \{ f \in C[a,b] : -f \in A \}$.

Then $\quad f$ crosses some lines

$$\Leftrightarrow \quad f \in A \cup B.$$

Hence $\quad C[a,b] \setminus Z = A \cup B$.

So we only need to show that A_n is nowhere dense $\forall n$, by proving

(1) A_n is closed $\forall n$, and

(2) $C[a,b] \setminus A_n$ is dense.

<u>Pf of (1)</u> Let $\{f_k\}$ be a seq. in A_n and

$$f_k \to f \text{ in } (C[a,b], d_\infty).$$

Since $f_k \in A_n$, $\exists$ $\alpha_k \in [-n,n]$ and $x_k \in [a,b]$

s.t. $\begin{cases} (f_k)_{-\alpha_k}(t) \le (f_k)_{-\alpha_k}(x_k), & \forall t \in (x_k - \frac{1}{n}, x_k) \\ (f_k)_{-\alpha_k}(t) \ge (f_k)_{-\alpha_k}(x_k), & \forall t \in (x_k, x_k + \frac{1}{n}) \end{cases}$ $(t \in [a,b])$

By passing to subsequence, we may assume

$$x_k \to x_0 \in [a,b]$$
$$\alpha_k \to \alpha_0 \in [-n,n].$$

Then $(f_k)_{-\alpha_k}(t) \le (f_k)_{-\alpha_k}(x_k)$ $\forall t \in (x_k - \frac{1}{n}, x_k)$

$\Leftrightarrow$ $f_k(t) - \alpha_k t \le f_k(x_k) - \alpha_k x_k$, $\forall t \in (x_k - \frac{1}{n}, x_k)$

Now $\forall t \in (x_0 - \frac{1}{n}, x_0)$, $\exists k_0 > 0$ s.t.

$t \in (x_k - \frac{1}{n}, x_k) \quad \forall\, k \geq k_0 \quad$ (Since $x_k \to x_0$)

Then $\quad f_k \to f$ in $(C[a,b], d_\infty)$, $\alpha_k \to \alpha_0$ we have

$$f(t) - \alpha_0 t \leq f(x_0) - \alpha_0 x_0 \quad \text{(by letting } k \to +\infty)$$

Since $t \in (x_0 - \frac{1}{n}, x_0)$ is arbitrary, we've proved

$$f_{-\alpha_0}(t) \leq f_{-\alpha_0}(x_0), \ \forall\, t \in (x_0 - \tfrac{1}{n}, x_0).$$

Similarly, one can prove

$$f_{-\alpha_0}(t) \geq f_{-\alpha_0}(x_0), \ \forall\, t \in (x_0, x_0 + \tfrac{1}{n}).$$

Hence $f \in A_n$. $\quad \therefore\ A_n$ is closed. ※

Pf of (2) $\quad$ Let $B^\infty_\varepsilon(f) \subset C[a,b]$ be a metric ball.

If $f \notin A_n$, then $B^\infty_\varepsilon(f) \cap \left(C[a,b] \setminus A_n\right) \neq \emptyset$.

If $f \in A_n$, by Weierstrass Approximation Theorem,

$\exists$ a polynomial p s.t. $\|p - f\|_\infty < \frac{\varepsilon}{3}$.

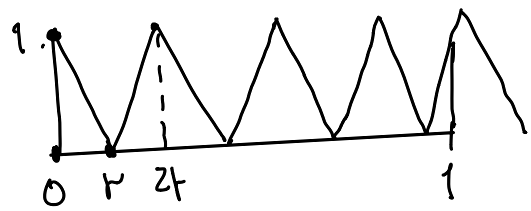
Define $\quad g(x) = p(x) + \frac{\varepsilon}{3}\varphi(x) \in C[a,b]$,

where φ is the restriction to $[a,b]$ of the

jig-saw function of period $2r$ satisfying $0 \leq \varphi \leq 1$

and slope of the graph of φ is $\pm\frac{1}{r}$ $(r>0)$

(except the finitely many non-differentiable points).



Then $\|g-f\|_\infty \leq \|g-p\|_\infty + \|p-f\|_\infty$

$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon$$

$\Rightarrow \quad g \in B_\varepsilon^\infty(f).$

Suppose that $g \in A_n$,

then $\exists\ x \in [a,b],\ \alpha \in [-n,n]$ s.t.

$$\begin{cases} g_{-\alpha}(t) \leq g_{-\alpha}(x), & t \in (x-\frac{1}{n}, x) \\ g_{-\alpha}(t) \geq g_{-\alpha}(x), & t \in (x, x+\frac{1}{n}) \end{cases}$$

If $\varphi(x) \in [0,\frac{1}{2}]$, then consider $\forall\ t \in (x-\frac{1}{n}, x)$,

$$p(t) + \frac{\varepsilon}{2}\varphi(t) - \alpha t \leq p(x) + \frac{\varepsilon}{2}\varphi(x) - \alpha x$$

$$\Rightarrow \quad \varphi(x) - \varphi(t) \geq \frac{2\alpha}{\varepsilon}(x-t) - \frac{2}{\varepsilon}(p(x)-p(t))$$

By the property of φ, $\exists t$ with $|t-x|<r$ (in both sides)

s.t. $\varphi(x)-\varphi(t) \leq -\frac{1}{2}$

Therefore, if $r<\frac{1}{n}$, then $\exists\, t\in(x-\frac{1}{n},x)$ s.t.

$$-\frac{1}{2} \geq \frac{2\alpha}{\varepsilon}(x-t)-\frac{2}{\varepsilon}(p(x)-p(t))$$

$$\Rightarrow \frac{1}{r} \leq \frac{4}{\varepsilon}(L+n) \quad \text{where } L=\text{Lip. const. of } p. \quad (|\alpha|\leq n)$$

It is a contradiction (as $r\to 0$).

Hence $\varphi(x)\in[\frac{1}{2},1]$.

Then consider $\forall\, t\in(x,x+\frac{1}{n})$

$$p(t)+\frac{\varepsilon}{2}\varphi(t)-\alpha t \geq p(x)+\frac{\varepsilon}{2}\varphi(x)-\alpha x$$

$$\Rightarrow \quad \varphi(t)-\varphi(x) \geq \frac{2\alpha}{\varepsilon}(t-x)-\frac{2}{\varepsilon}(p(t)-p(x))$$

By the property of φ, $\exists t$ with $|t-x|<r$ (in both sides)

s.t. $\varphi(t)-\varphi(x) \leq -\frac{1}{2}$

Therefore, if $r<\frac{1}{n}$, then $\exists\, t\in(x,x+\frac{1}{n})$ s.t.

$$-\frac{1}{2} \geq \frac{2\alpha}{\varepsilon}(t-x)-\frac{2}{\varepsilon}(p(t)-p(x))$$

$$\Rightarrow \frac{1}{r} \le \frac{4}{\varepsilon}(L+n)$$

It is again a contradiction.

Therefore $g \notin A_n$. And we've proved that

$$B_\varepsilon^\infty(f) \cap [C[a,b] \setminus A_n] \neq \emptyset.$$

This completes the proof of the Thm. #

Def: A function $f: [a,b] \to \mathbb{R}$ is said to be nowhere monotonic if $\exists$ no interval $[c,d] \subset [a,b]$ on which f is monotonic.

Cor: The set of continuous nowhere monotone functions is a residual set in $C[a,b]$, & hence dense in $C[a,b]$.

Pf: If $f \in C[a,b]$ is monotonic on some interval $[c,d]$, then $L_X \equiv b$ with $b \in (f(c), f(d))$ crosses f if $f(d) > f(c)$ (or $b \in (f(d), f(c))$ if $f(c) > f(d)$). If $f(c) = f(d)$, then $f \equiv$ const. on $[c,d]$. Clearly many lines cross f. Hence f monotonic on some interval $\Rightarrow$ $f \in C[a,b] \setminus Z$.

Since $C[a,b] \setminus Z$ is of 1^{st} category,

any subset of $C[a,b] \setminus Z$ is also of 1^{st} category.

$\Rightarrow$ set of cts functions monotonic on some interval is of 1^{st} category.

$\therefore$ set of cts nowhere monotonic functions is a residual. #

<u>Remark</u>: The Thm can be used to prove Thm 4.13 too.

Another application of Baire Category Theorem:

Thm 4.14 Every basis of an infinite dimensional Banach space consists of uncountably many vectors.

Pf: Let V be a Banach space.

Suppose on the contrary that V has a countable basis $\mathcal{B} = \{w_j\}_{j=1}^{\infty}$.

Then
$$V = \bigcup_{n=1}^{\infty} W_n,$$
where
$$W_n = \text{span}\{w_1, \cdots, w_n\}.$$

Claim 1: W_n has empty interior.

Pf: Since V is of infinite dimensional,
$$V \setminus W_n \neq \emptyset, \quad \forall n = 1, 2, \cdots$$
$$\Rightarrow \{v \in V : |v| = 1\} \setminus W_n \neq \emptyset, \quad \forall n = 1, 2, \cdots$$

$\therefore$ one can find $v_0 \in V \setminus W_n$ such that
$$|v_0| = 1.$$

Then $\forall\, w \in W_n$ and $\varepsilon > 0$,
$$w + \varepsilon v_0 \in B_\varepsilon(w) \cap (V \setminus W_n)$$

$\Rightarrow \quad B_\varepsilon(w) \cap (V \setminus W_n) \neq \phi$.

$\therefore$ W_n has empty interior.

<u>Claim 2</u>: W_n is closed, $\forall n = 1, 2, \cdots$

<u>Pf</u>: Let $\{v_j\}_{j=1}^{\infty}$ be a seq. in W_n

and converges to some $v_0 \in V$.

Note that
$$T: W_n \longrightarrow \mathbb{R}^n$$
$$\sum_{j=1}^n a_j w_j \longmapsto (a_1, \cdots, a_n)$$

is a vector space isomorphism.

And hence the norm in V, $\left|\sum_{j=1}^n a_j w_j\right|_V$

gives a norm on $\mathbb{R}^n$
$$\|(a_1, \cdots, a_n)\| = \left|\sum_{j=1}^n a_j w_j\right|_V.$$

Since any two norms on $\mathbb{R}^n$ are equivalent, (Ex!)

$\|(a_1, \cdots, a_n)\|$ is equivalent to standard Euclidean

norm $|(a_1, \cdots, a_n)| = \sqrt{a_1^2 + \cdots + a_n^2}$.

$\Rightarrow \exists\, C_1, C_2 > 0$ s.t.
$$|v|_V \leq C_1 |Tv| \leq C_2 |v|_V \qquad \forall\, v \in W_n$$

Since $v_\ell \to v_0$ in V, $\{v_\ell\}$ is Cauchy in $(V, |\cdot|_V)$.

$\therefore\ \forall \varepsilon > 0,\ \exists\, l_0 \geq 0$ s.t.

$$|v_l - v_k|_V < \varepsilon,\quad \forall\, l, k \geq l_0.$$

$$\Rightarrow \quad |Tv_l - Tv_k| \leq \frac{C_2}{C_1} |v_l - v_k|_V < \frac{C_2}{C_1}\varepsilon,\quad \forall l, k \geq l_0$$

$\Rightarrow \{Tv_l\}$ is Cauchy in $\mathbb{R}^n$ (with standard metric).

By completeness of $\mathbb{R}^n$, $\exists\ a^* = (a_1^*, \cdots, a_n^*) \in \mathbb{R}^n$

s.t. $\quad |Tv_l - a^*| \to 0$ as $l \to +\infty$.

Let $\quad v^* = T^{-1}a^* = \sum_{j=1}^{n} a_j^* w_j \in W_n,$

we have $\quad |v_l - v^*|_V \leq C_1 |Tv_l - a^*| \to 0$ as $l \to \infty$.

By uniqueness of limit, $v_0 = v^* \Rightarrow v_0 \in W_n$.

$\therefore$ W_n is closed. This proves Claim 2.

By Claims 1 & 2, W_n is nowhere dense and $V = \bigcup_{n=1}^{\infty} W_n$ is of 1^{st} Category. But V is complete, this is impossible. Hence any basis of V cannot be countable. ※