Cor 4.10 : Let (X, d) be complete. Suppose that $X = \bigcup_{n=1}^{\infty} E_n$ with E_n are closed subsets.

Then at least one of these E_n's has non-empty interior.

Pf: Since E_n is closed, E_n has empty interior implies E_n is nowhere dense. Hence if all E_n has empty interior, then $X = \bigcup_{n=1}^{\infty} E_n$ is of 1st category and hence X has empty interior by Baire Category Theorem. But interior of X is X itself which cannot be empty.

\#

Remark: This corollary implies that it is impossible to decompose a complete metric space into a countable union of nowhere dense sets.

Cor 4.11 A set of 1^{st} category in a complete metric space cannot be a residual set, and vice versa.

Pf: Let E be a set of 1^{st} category,

then $E = \bigcup_{n=1}^{\infty} E_n$ with E_n nowhere dense.

If E is also a residual set, then $X \setminus E$ is also of 1^{st} category, hence

$$X \setminus E = \bigcup_{n=1}^{\infty} E_n' \quad \text{with } E_n' \text{ nowhere dense.}$$

$$\Rightarrow \quad X = E \cup (X \setminus E) = \left(\bigcup_{n=1}^{\infty} E_n\right) \cup \left(\bigcup_{n=1}^{\infty} E_n'\right)$$

Taking closure, we have

$$X = \left(\bigcup_{n=1}^{\infty} \overline{E_n}\right) \cup \left(\bigcup_{n=1}^{\infty} \overline{E_n'}\right)$$

i.e. X is a countable union of close subsets with empty interiors.

This contradicts Cor 4.10. ※

Applications of Baire Category Theorem (to function spaces)

Thm 4.13 The set of all continuous, nowhere differentiable functions forms a residual set in $C[a,b]$ and hence dense in $C[a,b]$.

To prove the theorem, we need a lemma:

Lemma 4.2 Let $f \in C[a,b]$ be differentiable at x. Then it is Lipschitz continuous at x.

(It is clear near x. The main issue is for points not near x.)

Pf: By assumption, ($\forall \varepsilon > 0$, says $\varepsilon = 1$) $\exists \delta_0 > 0$ such that $\forall y \in (x-\delta_0, x+\delta_0) \setminus \{x\}$ (and $y \in [a,b]$)

$$\left| \frac{f(y)-f(x)}{y-x} - f'(x) \right| < 1,$$

$$\Rightarrow \quad |f(y)-f(x)| \le \left(1+|f'(x)|\right)|y-x|,$$

$$\forall\, y \in (x-\delta_0, x+\delta_0) \cap [a,b].$$

If $[a,b] \setminus (x-\delta_0, x+\delta_0) = \emptyset$, we are done.

If not, then for $y \in [a,b] \setminus (x-\delta_0, x+\delta_0)$,

$$|y-x| \ge \delta_0,$$

and hence

$$\begin{aligned} |f(y)-f(x)| &\le |f(y)| + |f(x)| \\ &\le 2\|f\|_\infty \\ &\le \frac{2\|f\|_\infty}{\delta_0}|y-x|. \end{aligned}$$

Let $L = \max\left\{ 1+|f'(x)|, \frac{2\|f\|_\infty}{\delta_0} \right\}$, we have

$$|f(y)-f(x)| \le L|y-x|, \quad \forall\, y \in [a,b]. \quad \#$$

Proof of Thm 4.13 : We only need to show the case $[a,b]=[0,1]$.

$\forall L>0$, define

$S_L = \{ f \in C[0,1] : f$ is Lip cts at some $x \in [0,1]$ with Lip constant $\leq L \}$

Claim 1: S_L is closed.

Pf: Let $\{f_n\}$ be a sequence in S_L with converges to some $f \in C[0,1]$ in d_∞ metric.

By definition of S_L, $\forall n \geq 1$,

$\exists x_n \in [0,1]$ such that

f_n is Lip. cts at x_n with Lip. const. $\leq L$.

i.e. $|f_n(y) - f_n(x_n)| \leq L|y - x_n|$, $\forall y \in [0,1]$.

We may assume that $x_n \to x^*$ for some $x^* \in [0,1]$ by passing to a subsequence. (The corresponding subsequence f_n is still convergent & $f_n \to f$ in d_∞.)

Then $|f(y) - f(x^*)| \leq |f(y) - f_n(y)| + |f_n(y) - f(x^*)|$

$\leq \|f - f_n\|_\infty + |f_n(y) - f_n(x_n)| + |f_n(x_n) - f(x^*)|$

$\leq \|f-f_n\|_\infty + L|y-x_n| + |f_n(x_n)-f_n(x^*)| + |f_n(x^*)-f(x^*)|$

$\leq \|f-f_n\|_\infty + L|y-x_n| + L|x_n-x^*| + \|f-f_n\|_\infty$

$\leq L|y-x^*| + 2\left(L|x^*-x_n| + \|f-f_n\|_\infty\right)$

Letting $n \to +\infty$, we have

$$|f(y)-f(x^*)| \leq L|y-x^*|, \quad \forall y \in [0,1].$$

$\Rightarrow \quad f \in S_L$. #

Claim 2: S_L is nowhere dense.

Pf: Let $f \in S_L$.

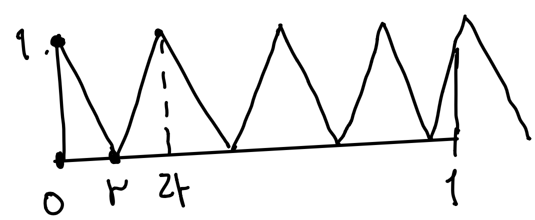
By Weierstrass Approximation Theorem,

$\forall \varepsilon > 0$, $\exists$ a polynomail p such that

$$\|f-p\|_\infty < \frac{\varepsilon}{2}$$

Let the Lip. constant of p be L_1.

For $r > 0$ ($r < 1$, not necessary rational), let



$\varphi(x)$ be the restriction to $[0,1]$

of the jig-saw function of period $2r$ satisfying $\varphi(0)=1$, $0 \le \varphi \le 1$, and slope of the graph of φ is $\pm \frac{1}{r}$ (except the finitely many non-differentiable points)

Then consider the function

$$g(x) = p(x) + \frac{\varepsilon}{2}\varphi(x) \in C[0,1].$$

Then
$$\|g-f\|_\infty \le \|p-f\|_\infty + \frac{\varepsilon}{2}\|\varphi\|_\infty$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

On the other hand,

$$\left|\frac{\varepsilon}{2}\varphi(y) - \frac{\varepsilon}{2}\varphi(x)\right| \le |g(y)-g(x)| + |p(y)-p(x)|$$

$$\Rightarrow \quad \frac{\varepsilon}{2}|\varphi(y)-\varphi(x)| \le |g(y)-g(x)| + L_1|y-x|$$

Note that $\forall x \in [0,1]$, $\exists y \in [0,1]$ near x such that

$$|\varphi(y)-\varphi(x)| = \frac{1}{r}|y-x|.$$

$$\Rightarrow \quad |g(y)-g(x)| \ge \left(\frac{\varepsilon}{2r} - L_1\right)|y-x|$$

Hence if we choose $r < \frac{\varepsilon}{2(L+L_1)}$, then

$\forall x \in [0,1]$, $\exists y \in [0,1]$ such that

$$|g(y)-g(x)| \geq \left(\frac{\varepsilon}{2r} - L_1\right)|y-x| > L|y-x| .$$

i.e. $\forall x \in [0,1]$, g is not Lip. cts at x with Lip const. L.

$\therefore \quad g \notin S_L$.

We have proved that $\forall f \in S_L$, $\forall \varepsilon > 0$

$$B_\varepsilon^\infty(f) \setminus S_L \neq \emptyset .$$

By claim 1, S_L is closed,

hence S_L is nowhere dense. #

<u>Final Step</u>:

Let $S = \{ f \in C[0,1] : f \text{ is differentiable at some } x \in [0,1] \}$.

Then by Lemma 4.12, $\forall f \in S$, $f \in S_N$ for some $N \in \mathbb{N}$.

(by enlarging the Lip. const. L given in the lemma)

$$\Rightarrow \quad S \subset \bigcup_{N=1}^{\infty} S_N$$

$\therefore$ S is of 1^{st} Category and has empty interior (since $(C[0,1], d_\infty)$ is complete).

$\Rightarrow$ Set of cts but nowhere differentiable functions on $[0,1]$

$\left(= \text{Complement of } S \text{ in } C[0,1]\right)$

is a residual set and dense in $C[0,1]$.

Remarks: (i) The Theorem and its proof provide no explicit example, not even a way to construct a cts nowhere differentiable function.

(ii) An explicit example was given by Weierstrass:

$$W(x) = \sum_{n=1}^{\infty} \frac{\cos(3^n x)}{2^n} \quad \text{on } \mathbb{R}$$

(it comes from Fourier series, actually Weierstrass provided a family.)