Pf of Implicit Function Theorem:

Define $\Phi : U^{\subset \mathbb{R}^n \times \mathbb{R}^m} \longrightarrow \mathbb{R}^n \times \mathbb{R}^m$

$$(x, y) \longmapsto (x, F(x,y))$$

where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$, $y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \in \mathbb{R}^m$.

Then $\Phi(x_0, y_0) = (x_0, 0)$.

Clearly Φ is C^k if F is C^k.

In fact

$$\Phi = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ F_1(x_1, \cdots, x_n, y_1, \cdots, y_m) \\ \vdots \\ F_m(x_1, \cdots, x_n, y_1, \cdots, y_m) \end{pmatrix}$$

$$\Rightarrow D\Phi = \left(\begin{array}{ccc|ccc} 1 & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & 1 & & & \\ \hline \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} & \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} & \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{array}\right)$$

Since $D_y F\Big|_{(x_0,y_0)} = \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & & \vdots \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{pmatrix}$ is invertible in $\mathbb{R}^m$,

$D\Phi\Big|_{(x_0,y_0)}$ is invertible in $\mathbb{R}^n \times \mathbb{R}^m$.

Applying Inverse Function Theorem, $\exists$ local C^1-inverse

$$\Psi = (\Psi_1, \Psi_2) : W^{\subset \mathbb{R}^n \times \mathbb{R}^m} \longrightarrow V,$$

with $(x_0, y_0) = \Psi(x_0, 0) = (\Psi_1(x_0,0), \Psi_2(x_0,0))$,

where W and V are open nbds. of $\Phi(x_0,y_0) = (x_0, 0)$ and (x_0, y_0) respectively, and is C^k when F is C^k.

By shrinking the nbds, we may assume V is of the form $V_1 \times V_2$, where V_1 open in $\mathbb{R}^n$ containing x_0; V_2 open in $\mathbb{R}^m$ containing y_0.

Now $\forall (x,z) \in W$,

$$\Phi(\Psi_1(x,z), \Psi_2(x,z)) = (x, z)$$

$$\parallel$$

$$(\Psi_1(x,z), F(\Psi_1(x,z), \Psi_2(x,z)))$$

$$\therefore \quad \begin{cases} x = \Psi_1(x,z) \\ z = F(\Psi_1(x,z), \Psi_2(x,z)) \end{cases}$$

$$\Rightarrow \qquad z = F(x, \Psi_2(x,z))$$

In particular, we can take $z=0$ & hence

$$F(x, \Psi_2(x,0)) = 0, \quad \forall\, x = \Psi_1(x,0) \in V_1.$$

$$\therefore \quad \varphi : V_1 \to V_2 : x \mapsto \Psi_2(x,0)$$

is the required map s.t.

$$\begin{cases} \varphi(x_0) = \Psi_2(x_0,0) = y_0, \\ F(x, \varphi(x)) = 0 \end{cases}$$

and is C^k when F is C^k. We've proved (1) & (2).

For (3), $D_y F$ is invertible in $V_1 \times V_2$

$$\Rightarrow \int_0^1 D_y F(x, y_1 + t(y_2 - y_1))\, dt \text{ is nonsingular}$$

for (x, y_1) & $(x, y_2) \in V_1 \times V_2$. (May assume V_2 is a ball)

Now if $\psi : V_1 \to V_2$ is another C^1-map

s.t. $F(x, \psi(x)) = 0$

then $0 = F(x,\psi(x)) - F(x,\varphi(x))$

$$= \left(\int_0^1 D_y F(x, \varphi(x) + t(\psi(x)-\varphi(x)))\,dt\right)(\psi(x)-\varphi(x))$$

$\int_0^1 D_y F(x, \varphi(x) + t(\psi(x)-\varphi(x)))\,dt$ nonsingular

$\Rightarrow \quad \psi(x) \equiv \varphi(x), \quad \forall\, x \in V_1$. #

<u>Remark</u>: Implicit Function Theorem and Inverse Function Theorem are in fact equivalent:

If $F: U \to \mathbb{R}^n$ as in assumption of the Inverse Function Theorem, then define

$$\tilde{F}(x,y): U \times \mathbb{R}^n \to \mathbb{R}^n \qquad (n+n \text{ to } n\text{-dim})$$
$$(x,y) \mapsto F(x)-y \qquad (\text{i.e. } m=n \text{ in the theorem})$$

which is C^1

Note that $\tilde{F}(x_0,y_0) = F(x_0) - y_0 = 0$, and

$$D_x\tilde{F}(x_0,y_0) = DF(x_0) \text{ is invertible}$$

$\Rightarrow$ $D\tilde{F}(x_0,y_0)$ is of full rank (rank $D\tilde{F}(x_0,y_0) = n$)

By Implicit Function Theorem, $\exists$ C^1-mapping $\varphi(y)$ near y_0 such that $\varphi(y_0) = x_0$ and $\tilde{F}(\varphi(y), y) = 0$.

(Note the different in the notations)

i.e. $F(\varphi(y)) - y = 0$ near y_0

$\therefore$ $x = \varphi(y)$ is the local inverse.

[Concrete examples are omitted since it should be given in advanced calculus already. A few explicit examples are given in Prof Chou's notes.]

§3.4 Picard–Lindelöf Theorem for Differential Equations

Let f be a function defined on

$$R = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b] \quad \text{where } (t_0, x_0) \in \mathbb{R}^2$$

and $a, b > 0$. We consider Cauchy Problem (Initial Value Problem)

$$(\text{IVP}) \quad \begin{cases} \dfrac{dx}{dt} = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

i.e. find a function $x(t)$ defined in a perhaps smaller interval $x : [t_0 - a', t_0 + a'] \to [x_0 - b, x_0 + b]$

such that

$$\begin{cases} x(t) \text{ is differentiable,} \\ x(t_0) = x_0, \text{ and} \\ \dfrac{dx}{dt}(t) = f(t, x(t)), \ \forall t \in [t_0 - a', t_0 + a'] \end{cases}$$

for some $0 < a' \leq a$.

eg 3.14 Consider $\begin{cases} \frac{dx}{dt} = 1+x^2 \\ x(0) = 0 \end{cases}$

Here $f(t,x) = 1+x^2$ is smooth on $[-a,a]\times[-b,b]$ for any $a,b>0$. However, the solution $x(t) = \tan t$ defined only on $(-\frac{\pi}{2}, \frac{\pi}{2})$. $\therefore$ Even for nice f, we may still have $a' < a$.

Recall :

(i) f defined in $R = [t_0-a, t_0+a]\times[x_0-b, x_0+b]$ satisfies the Lipschitz condition (uniform in t)

if $\exists\ L>0$ s.t. $\forall\ (t,x_1), (t,x_2) \in R$,

$$|f(t,x_1) - f(t,x_2)| \le L|x_1 - x_2| .$$

(ii) In particular, $f(t,\cdot)$ is Lip. cts in x, $\forall\ t\in[t_0-a, t_0+a]$.

(iii) L is called a Lipschitz constant.

(iv) If L is a Lip. constant for f, then any $L' > L$ is also a Lip. constant.

(v) Not all cts. functions satisfy the Lip. condition.

eg: $f(t,x) = t x^{1/2}$ is cts, but not Lip. near 0.

(vi) If $R = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$ and

$f(t,x) : R \to \mathbb{R}$ is C^1,

then $f(t,x)$ satisfies the Lip. condition:

in fact, for some $y \in [x_0 - b, x_0 + b]$,

$$|f(t,x_2) - f(t,x_1)| = \left| \frac{\partial f}{\partial x}(t,y)(x_2 - x_1) \right|$$

Hence

$$|f(t,x_2) - f(t,x_1)| \le L |x_2 - x_1|,$$

for $L = \max\left\{ \left| \frac{\partial f}{\partial x}(t,x) \right| : (t,x) \in R \right\}$.

Thm 3.10 (Picard–Lindelöf Theorem)

(Fundamental Theorem of Existence and Uniqueness of Differential Equations)

Let f be continuous function on $R = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$ $((t_0, x_0) \in \mathbb{R}^2,\ a, b > 0)$ satisfies the Lipschitz condition on R (uniform in t). Then $\exists\ a' \in (0, a]$ and $x \in C^1[t_0 - a', t_0 + a']$ such that

$$x_0 - b \le x(t) \le x_0 + b, \quad \forall\ t \in [t_0 - a', t_0 + a']$$

and solving the Cauchy Problem (IVP)

Furthermore, x is the unique solution in $[t_0 - a', t_0 + a']$.

Note: One will see in the following proof that a' can be taken to be any number satisfying

$$0 < a' < \min\left\{a, \frac{b}{M}, \frac{1}{L}\right\}$$

where

$$M = \sup\{|f(t, x)| : (t, x) \in R\} \ \&$$

$$L = \text{Lip. const. for } f.$$

Prop 3.11: Setting as in Thm 3.10, every solution x of (IVP) from $[t_0-a', t_0+a']$ to $[x_0-b, x_0+b]$ satisfies the equation

$$\boxed{x(t) = x_0 + \int_{t_0}^{t} f(t, x(t))\,dt} \quad (3.7)$$

Conversely, every $x(t) \in C[t_0-a', t_0+a']$ satisfying (3.7) is C^1 and solves (IVP).

<u>Pf</u>: Obvious, by Fundamental Theorem of Calculus.