Ch3 The Cartraction Mapping Principle

Pf: (a) Let ECX, & E complete.

Suppose
$$\{x_n\} \in E$$
 with $x_n \to x$ in \mathbb{X} .
By note, $1x_n \notin i$ is a Cauchy seq. in E
Then completeness of $E \Rightarrow x_n \to z \in E$.
Uniqueness of limit $\Rightarrow x = z \in E$
 $-2 \in E$ closed.

eg3.1: • (IR, standard) is complete
• [a,b], (-os, b], [a, os) complete
• [a,b) (b finite) not complete (-:
$$x_n=b-f_n > b \notin [0,b)$$
)
• @ is not complete.

3.5 Appendix : Completion of a Metric Space

Def: A metric space
$$(X,d)$$
 is said to be isometrically
embedded in metric space (\overline{Y}, p) if
 $\exists a$ mapping $\overline{\Phi} : X \to Y \quad s.t.$
 $d(x,y) = p(\overline{\Phi}(x), \overline{\Phi}(y))$.

Def: Let
$$(X,d)$$
 and $(\overline{Y}, \overline{P})$ be metric spaces.
We call $(\overline{Y}, \overline{P})$ a completion of (\overline{X}, d)
 $\overrightarrow{\mathcal{A}}$ (1) $(\overline{Y}, \overline{P})$ is complete.
(2) \exists isometric embedding $\underline{\Phi}^{\pm}(\overline{X}, d) \rightarrow (\overline{Y}, \overline{P})$
such that the closure $\overline{\Phi}(\overline{X}) = \overline{Y}$.

$$\underline{eg}: (Y, \rho) = (\mathbb{R}, \text{standard}), X = \Theta \subset \mathbb{R}$$

$$(X, d) = (\Theta, \text{induced notic})$$
Then $\cdot (\mathbb{R}, \text{standard})$ is complete;
$$\overline{\Phi} = (\Theta, \text{induced metric}) \longrightarrow (\mathbb{R}, \text{standard})$$

$$\overset{\vee}{B} \longrightarrow \overset{\vee}{B}$$

$$\cdot X = \overline{Q} = \mathbb{R} (\Theta \text{ is dense in } \mathbb{R})$$

$$\xrightarrow{\vee}{\overline{\Phi}(\Theta)}$$

Ef (Sketch of Proof)
Let (X, d) be metric space.
Let
$$\mathcal{E} = \{ ix_n \} \subset X = ix_n \}$$
 Cauchy soguence $\}$
Define equivalent relation ~ on \mathcal{E} by
 $\{ x_n \} \sim \{ y_n \} \iff d(x_n, y_n) \Rightarrow 0 \text{ as } n \Rightarrow \infty$
Let $\widetilde{\mathcal{E}} = \mathcal{E}/\mathcal{L}$ the quotient space.

Define
$$\widehat{d} = \widehat{\mathbb{E}} \times \widehat{\mathbb{E}} \to \mathbb{IR}$$
 by the following:
For $\widehat{X} = equi. dass [i \times n S]$
 $\widehat{Y} = equi. dass [i \times n S]$,
 $\widehat{d}(\widehat{X}, \widehat{Y}) = \lim_{n \to \infty} d(x_n, y_n)$
Then \widehat{d} is well-defined and is a
metric on $\widehat{\mathbb{E}}$.
One then proves $(\widehat{\mathbb{E}}, \widehat{d})$ is complete.
 $\overline{\Phi} = (\widehat{X}, \widehat{d}) \longrightarrow (\widehat{\mathbb{E}}, \widehat{d})$ defined by
 $\widehat{X} \longmapsto [i \times x, x, \dots, S]$
is an isometric embedding.
And one can show that
 $\overline{\Phi}(\widehat{X}) = \widehat{\mathbb{E}}$.
 $\widehat{\Phi}(\widehat{X}) = \widehat{\mathbb{E}}$.
Two metric spaces $(\widehat{X}, d), (\widehat{X}', d')$ one
called isometric is $\overline{\Xi}$ bijective isometric
embedding from (\widehat{X}, d) onto (\widehat{X}', d') .

Def =

\$3,2 The Contraction Mapping Principle Ref: (1) Let (X, d) le a metric space. A map $T: (X,d) \rightarrow (X,d)$ is called a contraction if \exists constant $\gamma \in (0, 1)$, such that $d(Tx,Ty) \leq \forall d(x,y), \forall x,y \in \mathbb{X}$. (2) A point XEX is called a fixed point of T if Tx = x(Usually we write Tx instead of T(x).)

Thm 3.3 (Contraction Mapping Principle) Every contraction in a complete metric space adrit à rinque fixed point. (This is also called the Banach Fixed Point Thm)

$$\frac{Pf}{2}: \frac{2\ln i guenoss}{2}: Suppose \times x y are fixed pts.}$$
of T. Then
$$d(x,y) = d(Tx, Ty) \quad (x, y are fixed by T)$$

$$\leq x d(x, y) \quad fer \quad some \quad x \in (0, 1).$$

$$(T \quad cartraction)$$

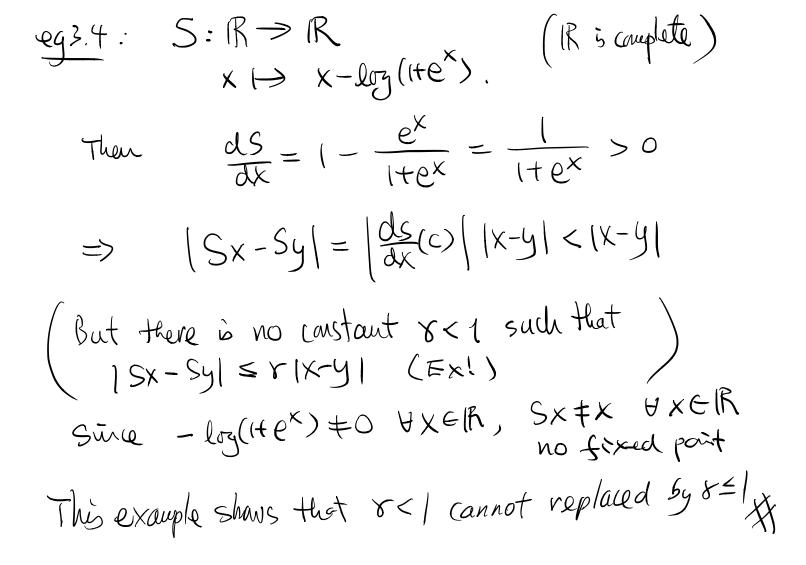
$$\Rightarrow d(x, y) = 0 \Rightarrow \quad x = y.$$

Existence: Let
$$x_0 \in \mathbb{X}$$
.
Define $\{x_n\}_{n=1}^{\infty}$ by $x_n = Tx_{n-1}$, $f_{n} n = 1, 2$.
Then $x_n = Tx_{n-1} = T(Tx_{n-2}) = T^2 x_{n-2}$
 $= --- = T^n x_0$.

For any $n \ge N$, $d(x_n, x_N) = d(Tx_0, T^N x_0) = d(T^{(n-N)+N} x_0, T^N x_0)$ $= d(T(T^{(n-N)+N+1} x_0), T(T^{N-1} x_0))$ $\leq \gamma d((T^{(n-N)+N+1} x_0, T^{N-1} x_0))$ (where $\gamma \in (0, 1)$ is the constant s.t. $d(Tx, Ty) \le r d(x, y), \forall x y \in X$)

۲.... $\leq \gamma^{N} d(T^{(n-N)} x_{0} x_{0})$ $\leq \gamma^{N} \int d(T^{(n-N)}_{x_{0}}, T^{(n-N)-1}_{x_{0}}) + d(T^{(n-N)-1}_{x_{0}}, T^{(n-N)-2}_{x_{0}})$ $+ \cdots + d(Tx_0, x_0)$ $\leq \gamma^{N} | d(Tx_{o}, x_{o}) + rd(Tx_{o}, x_{o}) + \cdots$ + $\gamma^{(n-N)-2} d(Tx_{o}, x_{0}) + \gamma^{(n-N)-1} d(Tx_{o}, x_{0})$ $= \gamma^{N} \int [+\gamma + \cdots + \gamma^{(n-N)-1}] d(T_{X_{0}} X_{0})$ $< \frac{\gamma''}{1} d(Tx_0, x_0)$ Therefore, 42>0, if N>0 is chosen s.t. $\frac{\chi^{N}}{1-\chi} d(Tx_{0}, x_{0}) < \xi$ we have $\forall n, m \ge N$ $d(x_n, x_m) \leq d(x_n, x_N) + d(x_N, x_m)$ く ディナディーを. $\therefore \{x_n\}$ is a Cauchy seq. $\hat{m}(X,d)$.

By completeness of (X, d), I XEX s.f. Note that a contraction is always contained (Ex!) $K = \lim_{n \to \infty} X_n = \lim_{n \to \infty} T_{X_{n-1}} = T \lim_{n \to \infty} X_{n-1} = T_X.$ we have · · × is a fixed point of T. × (Caution: (0,1] is not eg 3.3 T: (0,1] → (0,1] complete) $X \mapsto \frac{X}{z}$ Chapterly $|Tx - Ty| = \frac{1}{2}|x - y| \quad x = \frac{1}{2} < 1$:. Tis a contraction. However, if XE(0, 1] is a fixed point of T, $TX=X \Leftrightarrow X=X \Leftrightarrow X=0 \notin (0,1],$ then - Thas no fixed paint on (0,1]. This example shows that "completeness" is necessary in the Contraction Mapping Principle.

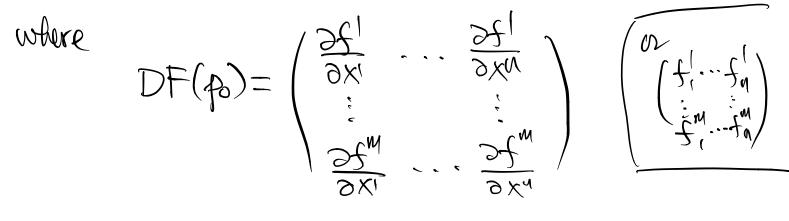


§3.3 The Inverse Function Theorem
Notation: Let
$$F = U \subset \mathbb{R}^n \to \mathbb{R}^m$$
 be differentiable
at a paint p in an open set U of \mathbb{R}^n . We
write $F = \begin{pmatrix} f \\ g \\ f \end{pmatrix} \in \mathbb{R}^m$,
where $f^i = f^i(x_1^j, \dots, x^n) = U \to \mathbb{R}$, $\forall i = 1, \dots, m$.
Then F differentiable at $p_0 = \begin{pmatrix} x_0^j \\ x_0^n \end{pmatrix} \in U \subset \mathbb{R}^n$

$$F(p_0+z) - F(p_0) = DF(p_0)z + o(z)$$

$$\forall z = \begin{pmatrix} z' \\ \vdots \\ z'' \end{pmatrix} \text{ sufficiently small,}$$

$$(1z1 \text{ small})$$



i.e. $(DF(p_0)Z)^i = \sum_{j=1}^n \frac{\partial f^i}{\partial X^j}(p_0)Z^j$ $Y^i = J^m$.