MATH3060 HW3 Due date: Oct 16, 2020 (at 12:00 noon)

(1) Sketch the metric ball of radius 1 centered at 0 in $\mathbb{R}^2$ for the metrics d_1, d_2 and d_∞ on $\mathbb{R}^2$.

(2) Show that for any $\alpha \in \mathbb{R}$, the set

$$\{ f \in C[a,b] : f(x) \geq \alpha, \ \forall x \in [a,b] \}$$

is closed in $(C[a,b], d_\infty)$.

(3)

(a) Let $l_1 = \{ x = (x_1, x_2, \cdots) : \sum_{i=1}^{\infty} |x_i| < \infty, \ x_i \in \mathbb{R} \}$.

Show that $d_1(x,y) = \sum_{i=1}^{\infty} |x_i - y_i|$ is a metric on l_1

(b) Let $l_2 = \{ x = (x_1, x_2, \cdots) : \sum_{i=1}^{\infty} |x_i|^2 < \infty, \ x_i \in \mathbb{R} \}$.

Show that $d_2(x,y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^2 \right)^{1/2}$ is a metric on l_2

(c) Let $l_\infty = \{ x = (x_1, x_2, \cdots) : \sup_i |x_i| < \infty, \ x_i \in \mathbb{R} \}$.

Show that $d_\infty(x,y) = \sup_i |x_i - y_i|$ is a metric on l_∞

(d) Show that as sets,

$$l_1 \subset l_2 \subset l_\infty .$$

(4) Let $C^1[a,b] = \{f \in C[a,b] : f$ is continuous differentiable on $[a,b]\}$.

Define $\forall f, g \in C^1[a,b]$

$$d(f,g) = \|f-g\|_\infty + \|f'-g'\|_\infty .$$

Show that d is a metric on $C^1[a,b]$. Furthermore,

Is $f_k(x) = \dfrac{\sin(kx)}{k}$, $k=1,2,\cdots$, a convergence sequence

in $(C^1[0,1], d)$?

(5) Let (X_1, d_1) and (X_2, d_2) be two metric spaces.

Define $d : (X_1 \times X_2) \times (X_1 \times X_2) \to \mathbb{R}$ by

$$d(u,v) = d_1(x_1, y_1) + d_2(x_2, y_2),$$

$\forall\ u = (x_1, x_2)$ and $v = (y_1, y_2) \in X_1 \times X_2$.

(a) Show that d is a metric on $X_1 \times X_2$.

(It is called the product metric)

(b) Show that if G_1 is an open set of (X_1, d_1) and G_2 is an open set of (X_2, d_2), then $G_1 \times G_2$ is an open set of $(X_1 \times X_2, d)$.

(End)