§2.2 Open and Closed Sets

<u>Def</u>: Let (X, d) = metric space

- A set $G \subset X$ is called an <u>open set</u> if $\forall\, x \in G$, $\exists\ \underline{\varepsilon > 0}$ s.t. $B_\varepsilon(x) = \{y : d(x,y) < \varepsilon\} \subset G$.

(The number $\varepsilon > 0$ may vary depending on x.)

- We also define the <u>empty set</u> $\emptyset$ to be an <u>open set</u>.

<u>Prop 2.4</u>: Let (X, d) be a metric space. We have

(a) X and $\emptyset$ are open sets.

(b) Arbitrary union of open sets is open; if G_α, $\alpha \in A$, is a collection of open sets then $\bigcup_{\alpha \in A} G_\alpha$ is an open set.

(c) Finite intersection of open sets is open: if $G_1, \cdots, G_N$ are open sets, then $\bigcap_{j=1}^{N} G_j$ is an open set.

Pf: (a) Clear

(b) Let $x \in \bigcup_{\alpha \in A} G_\alpha$

$\Rightarrow x \in G_\alpha$ for some $\alpha \in A$.

$\Rightarrow \exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset G_\alpha$ (since G_α open)

$\Rightarrow B_\varepsilon(x) \subset \bigcup_{\alpha \in A} G_\alpha$. ※

(c) Let $x \in \bigcap_{j=1}^{N} G_j$

$\Rightarrow x \in G_j, \ \forall j = 1, \cdots, N$

$\Rightarrow \exists \varepsilon_j > 0$ s.t. $B_{\varepsilon_j}(x) \subset G_j, \ \forall j = 1, \cdots, N$.

Let $\varepsilon = \min\{\varepsilon_1, \cdots, \varepsilon_N\} > 0$. Then

$B_\varepsilon(x) \subset B_{\varepsilon_j}(x) \subset G_j, \ \forall j = 1, \cdots, N$

$\Rightarrow B_\varepsilon(x) \subset \bigcap_{j=1}^{N} G_j$ ※

Def: Let (X, d) be a metric space.
A set $F \subset X$ is called a closed set if the complement $X \setminus F$ is an open set.

Prop 2.5 Let (X, d) be a metric space. We have

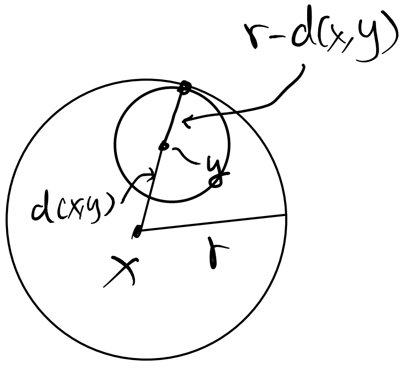
(a) X and $\emptyset$ are closed sets.

(b) Arbitrary intersection of closed sets is closed:
if F_α, $\alpha \in A$, are closed sets, then $\bigcap_{\alpha \in A} F_\alpha$ is closed.

(c) Finite union of closed sets is closed:
if $F_1, \cdots, F_N$ are closed sets, then $\bigcup_{j=1}^{N} F_j$ is closed.

Note: Prop 2.4 & 2.5 $\Rightarrow$ X & $\emptyset$ are both open & closed.

eg 2.10 (1) Every metric ball $B_r(x) = \{ y \in X : d(y,x) < r \}$ $(r > 0)$ is an open set.



Pf: $\forall\, y \in B_r(x)$

Then $\varepsilon = r - d(x,y) > 0$

& $\forall\, z \in B_\varepsilon(y)$,

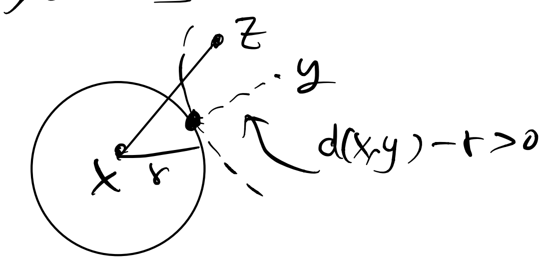
$$d(z,x) \le d(z,y) + d(y,x) < \varepsilon + d(y,x) = r$$

$\Rightarrow$ $B_\varepsilon(y) \subset B_r(x)$ #

(2) The set $E = \{ y \in X : d(y,x) > r \}$ (for a fixed $x \in X$)

is open and hence

$X \setminus E = \{ y \in X : d(y,x) \le r \}$ is closed.



Pf: $\forall y \in E$

Then $\varepsilon = d(x,y) - r > 0$

$\forall z \in B_\varepsilon(y)$,

$$d(z,x) \ge d(x,y) - d(z,y) \quad \text{(triangle ineq.)}$$
$$> d(x,y) - (d(x,y) - r)$$
$$= r$$

$\therefore \quad B_\varepsilon(y) \subset E$ ※

Note: We usually write (Confusing notation here, may not equal to the "closure" of $B_r(x)$ in a general metric space!)

$$\bar{B}_r(x) = \overline{B_r(x)} = \{ y \in X : d(y,x) \le r \}$$

the closed ball of radius r centered at x.

(3) Since $B_r(x)$ & $E = \{ y \in X : d(x,y) > r \}$ are open,

$B_r(x) \cup E$ is open.

$\Rightarrow \quad X \setminus (B_r(x) \cup E) = \{ y \in X : d(x,y) = r \}$

is closed.

In particular, $E = \{ y \in X : d(y,x) > 0 \}$ is open

$\Rightarrow \quad \{x\} = X \setminus E$ is <u>closed</u> (in any metric space)

[Note: $\{x\}$ may not be open (unless $\exists\, \varepsilon_0 > 0$ s.t. $B_{\varepsilon_0}(x) = \{x\}$.)]

<u>eg 2.11</u> $B_{\frac{1}{n}}(x)$, $n = 1, 2, \cdots$, are open sets.

<u>Claim</u>: $\bigcap_{n=1}^{\infty} B_{\frac{1}{n}}(x) = \{x\}$ (closed, may not be open)

($\therefore$ infinite intersection of open sets may not be open)

<u>Pf of claim</u>: $\forall y \in \bigcap_{n=1}^{\infty} B_{\frac{1}{n}}(x) \Rightarrow y \in B_{\frac{1}{n}}(x)$, $\forall n = 1, 2, \cdots$

$\Rightarrow \quad d(y,x) < \frac{1}{n}$, $\forall n = 1, 2, \cdots$

$\Rightarrow \quad d(y,x) = 0$.

$\Rightarrow \quad y = x$ $\quad$ ※

<u>eg 2.13</u> $X = C[a,b]$ with $d_\infty(f,g) = \|f-g\|_\infty = \sup_{x \in [a,b]} |f-g|(x)$

let $E = \{ f \in C[a,b] : f(x) > 0, \ \forall x \in [a,b] \} \subset X$

$\forall f \in E$, f is positive, cts on the closed & bounded interval $[a,b]$, therefore $\exists\, m > 0$ s.t.

$$f(x) \geq m > 0 \quad , \forall x \in [a,b].$$

Consider $B^{\infty}_{\frac{m}{2}}(f) = \{ g \in C[a,b] : d_\infty(g,f) < \frac{m}{2} \}$

$\forall\, g \in B^{\infty}_{\frac{m}{2}}(f)$, we have $\forall x \in [a,b]$

$$\begin{aligned} g(x) &= [g(x) - f(x)] + f(x) \\ &\geq f(x) - \|g-f\|_\infty \\ &> f(x) - \frac{m}{2} \geq m - \frac{m}{2} = \frac{m}{2} > 0 \end{aligned}$$

$\therefore$ $g \in E$ & hence $B^{\infty}_{\frac{m}{2}}(f) \subset E$.

$\therefore$ E is open in $(C[a,b], d_\infty)$.

Similarly, one can show that $\forall \alpha \in \mathbb{R}$

$$\{ f \in C[a,b] : f(x) > \alpha, \ \forall x \in [a,b] \}$$

$$\{ f \in C[a,b] : f(x) < \alpha, \ \forall x \in [a,b] \}$$

are open in $(C[a,b], d_\infty)$.

And $\{f \in C[a,b] : f(x) \geq \alpha, \ \forall x \in [a,b]\}$

$\{f \in C[a,b] : f(x) \leq \alpha, \ \forall x \in [a,b]\}$

are closed in $(C[a,b], d_\infty)$ (Ex!)

(Caution : $C[a,b] \setminus \{f \in C[a,b] : f(x) \geq \alpha, \forall x \in [a,b]\}$
$\neq \{f \in [a,b] : f(x) < \alpha, \forall x \in [a,b]$)