Def: Let d and ρ be 2 metrics defined on X.

(1) We call ρ is <u>stronger than</u> d or d is <u>weaker than</u> ρ, if $\exists\, C>0$ s.t.

$$d(x,y) \le C\rho(x,y), \quad \forall x,y \in X.$$

(2) They are <u>equivalent</u> if ρ is stronger and weaker than d. i.e. $\exists\, C_1, C_2 > 0$ s.t.

$$d(x,y) \le C_1\rho(x,y) \le C_2 d(x,y), \quad \forall x,y \in X.$$

(or $c_1 d(x,y) \le \rho(x,y) \le c_2 d(x,y), \ \forall x,y \in X$)

Prop: (1) If ρ is stronger than d, then $\{x_n\}$ converges in (X,ρ) implies $\{x_n\}$ converges in (X,d), and hence the same limit.

(2) If ρ is equivalent to d, then $\{x_n\}$ converges in (X,ρ) if and only if $\{x_n\}$ converges in (X,d).

(3) "equivalent" of metrics defined above is an equivalent relation.

(Pf: Easy ex.)

eg: On $\mathbb{R}^n$, $\begin{cases} d_1(x,y) = \sum_i |x_i - y_i| \\ d_2(x,y) = \left(\sum_i |x_i - y_i|^2\right)^{1/2} \\ d_\infty(x,y) = \max_i |x_i - y_i| \end{cases}$

Check: (i) $d_2(x,y) \leq \sqrt{n}\, d_\infty(x,y) \leq \sqrt{n}\, d_2(x,y)$

(ii) $d_1(x,y) \leq n\, d_\infty(x,y) \leq n\, d_1(x,y)$.

Therefore, d_1, d_2, & d_∞ are equivalent metrics on $\mathbb{R}^n$.

eg: $X = C[a,b]$, $\begin{cases} d_1(f,g) = \int_a^b |f-g| \\ d_\infty(f,g) = \max_{[a,b]} |f-g| \end{cases}$

Then clearly

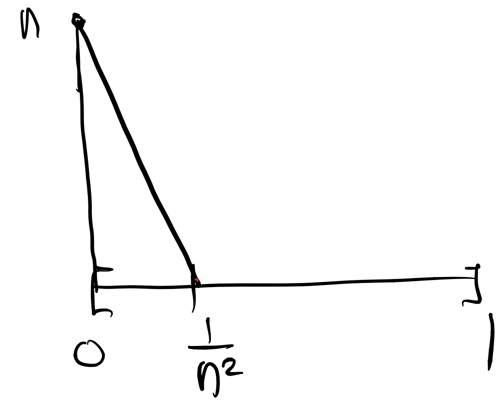
$$d_1(f,g) \leq (b-a)\, d_\infty(f,g), \ \forall f,g \in C[a,b].$$

$\therefore$ d_∞ is stronger than d_1.

However, it is impossible to find $C>0$ s.t.

$$d_\infty(f,g) \leq C\, d_1(f,g), \ \forall f,g \in C[a,b].$$

Pf: Define f_n on $[a,b]=[0,1]$



$$f_n(x) = \begin{cases} -n^3x+n, & x\in[0,\frac{1}{n^2}] \\ 0, & x\in(\frac{1}{n^2},1] \end{cases}$$

Then $d_1(f_n,0) = \int_0^1 |f_n| = \frac{1}{2n} \to 0$ as $n\to+\infty$

& $d_\infty(f_n,0) = \max_{[0,1]} |f_n(x)| = n$

$\therefore$ $n = d_\infty(f_n,0) \le C' d_1(f_n,0) = \frac{C}{2n}$, $\forall n$

which is impossible.

$\therefore$ d_1 is <u>not</u> stronger than d_∞.

Therefore d_1 & d_∞ are <u>not</u> equivalent.

<u>Def</u>: Let $f:(X,d)\to(Y,\rho)$ be a mapping between 2 metric spaces, and $x\in X$. We call f is <u>continuity at</u> x if

$f(x_n)\to f(x)$ in (Y,ρ) whenever $x_n\to x$ in (X,d).

It is <u>continuous</u> on a set $E\subset X$ if it is continuous at every point of E.

Prop 2.2 Let $f:(X,d)\to(Y,\rho)$ be a mapping between 2 metric spaces, and $x_0\in X$. Then

f is continuous at x_0

$\Longleftrightarrow \begin{cases} \forall \varepsilon>0,\ \exists\delta>0 \text{ such that} \\ \rho(f(x), f(x_0))<\varepsilon,\ \forall x \text{ with } d(x,x_0)<\delta. \end{cases}$

(Pf: Ex!)

Prop 2.3: Let $f:(X,d)\to(Y,\rho)$ &

$g:(Y,\rho)\to(Z,m)$

are mappings between metric spaces.

(a) If f is continuous at x & g is continuous at $f(x)$, then $g\circ f:(X,d)\to(Z,m)$ is continuous at x.

(b) If f is cts in X and g is cts in Y, then $g\circ f$ is cts in X.

(Pf: Easy)

Eg: Let (X, d) be a metric space, $A \subset X$, $A \neq \emptyset$.

Define $\rho_A : X \to \mathbb{R}$ by

$$\rho_A(x) = \inf_{y \in A} d(y, x)$$

(distance from x to the subset A).

Claim: $|\rho_A(x) - \rho_A(y)| \le d(x,y)$, $\forall x, y \in X$.

Pf of claim: For fixed $x, y \in X$.

By defn. of $\rho_A(y)$,

$$\forall \varepsilon > 0, \ \exists z \in A \text{ s.t. } \rho_A(y) + \varepsilon > d(z, y)$$

Hence, $\rho_A(x) \le d(z, x) \le d(x,y) + d(y,z)$
$$< d(x,y) + \rho_A(y) + \varepsilon$$

$$\Rightarrow \quad \rho_A(x) - \rho_A(y) < d(x,y) + \varepsilon.$$

Interchanging the roles of x & y

$$\rho_A(y) - \rho_A(x) < d(x,y) + \varepsilon$$

Therefore $|\rho_A(x) - \rho_A(y)| < d(x,y) + \varepsilon$.

Since $\varepsilon > 0$ is arbitrary, $|\rho_A(x) - \rho_A(y)| \le d(x,y)$

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By claim, $d(x_n, x) \to 0 \Rightarrow \rho_A(x_n) \to \rho_A(x)$

$\therefore \quad \rho_A : (X, d) \to \mathbb{R}$ is cts.

(In fact, ρ_A is "Lipschitz continuous".)

This example shows that there are "many" cts functions on a metric space.

<u>Notation</u>: Usually, we use the following notations

$$d(x, F) = \inf\{ d(x,y) : y \in F \}$$

$$d(E, F) = \inf\{ d(x,y) : x \in E, y \in F \}.$$

for subsets E & F.