MATH3060 HW2 Due date: Oct 6, 2020 (at 12:00 noon)

1. A finite Fourier series is of the form

$$a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx).$$

A trigonometric polynomial is of the form

$$p(\cos x, \sin x)$$

where $p(x,y)$ is a polynomial of 2 variables x, y.
Show that a function is a trigonometric polynomial if and only if it is a finite Fourier series.

2. Let f be a C^∞ 2π-periodic function on $[-\pi, \pi]$.
Show that the Fourier coefficients

$$|a_n| = o\left(\frac{1}{n^k}\right) \text{ and } |b_n| = o\left(\frac{1}{n^k}\right)$$

as $n \to \pm\infty$ for every k.

3. Let f, g and $h \in R[a,b]$. Show that

$$\|f-g\|_2 \leq \|f-h\|_2 + \|h-g\|_2.$$

When does the equality sign hold?

4. Let f, g be 2π-periodic functions integrable on $[-\pi, \pi]$.

Show that

$$\int_{-\pi}^{\pi} fg = 2\pi\, a_0(f)\, a_0(g) + \pi \sum_{n=1}^{\infty} \left[a_n(f)\, a_n(g) + b_n(f)\, b_n(g) \right]$$

where a_0, a_n, b_n are corresponding Fourier coefficients.

5. Using Parseval's Identity for $f(x) = x$ on $[-\pi, \pi]$ to show the famous equality

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(End)