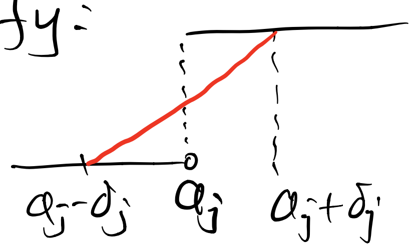
Thm 1.16 For 2π-periodic (real) function f integrable (Riemann) on $[-\pi, \pi]$,

$$\lim_{n\to\infty} \| S_n f - f \|_2 = 0 .$$

i.e. the n-partial sum of the Fourier series of f converges to f in L^2-sense.

Pf: Step 1: $\forall \varepsilon > 0$, $\exists$ a 2π-periodic Lip. cts function g s.t. $\|f-g\|_2 < \varepsilon/2$.

(Ex: Hint: find step function approximating f as before, then modify:)



Step 2: Completion of the proof.

Applying Thm 1.7 to the function g in Step 1:

$$\exists N > 0 \text{ s.t. } \quad \| g - S_N g \|_\infty < \frac{\varepsilon}{2\sqrt{2\pi}}$$

uniform convergence

Thus $\|g - S_N g\|_2 = \sqrt{\int_{-\pi}^{\pi} (g - S_N g)^2} \le \sqrt{2\pi \|g - S_N g\|_\infty^2}$

$= \frac{\varepsilon}{2}$.

By Cor 1.15,

$\|f - S_N f\|_2 \le \|f - S_N g\|_2$ ($S_N g$ is of the form $\alpha_0 + \sum_{k=1}^{N} (\alpha_k \cos kx + \beta_k \sin kx)$)

$\le \|f - g\|_2 + \|g - S_N g\|_2$ (Ex!) (HW2 Q3)

$< \underbrace{\frac{\varepsilon}{2}}_{\text{(step 1)}} + \frac{\varepsilon}{2} = \varepsilon$

Finally, since $E_N \subset E_n$, $\forall n \ge N$
(↑ have more generators),

we have $\forall n \ge N$,

$\|f - S_n f\|_2 \le \|f - S_N f\|_2$ (by Cor 1.15 over the subsp. E_n)

$< \varepsilon$.

$\therefore \lim_{n \to \infty} \|S_n f - f\|_2 = 0$ #

Cor 1.17 (a) Suppose that f_1 & f_2 are 2π-periodic integrable functions on $[-\pi, \pi]$ with the same Fourier series. Then $f_1 = f_2$ almost everywhere (i.e. $f_1 = f_2$ except a set of measure zero.)

(b) Suppose that f_1 and f_2 are 2π-periodic continuous functions with the same Fourier series. Then $f_1 = f_2$.

Recall: A set E is said to be of measure zero if $\forall \varepsilon > 0$, $\exists$ countably many intervals $\{I_k\}$ s.t.

$$E \subset \bigcup_k I_k \quad \& \quad \sum_k |I_k| < \varepsilon.$$

Pf: (a) Let $f = f_1 - f_2$, then $a_n(f) = b_n(f) = 0$

$$\Rightarrow S_n f = 0 \quad \forall n \geq 0$$

Hence $$\lim_{n\to\infty} \|S_n f - f\|_2 = 0$$

$$\Rightarrow \quad \|f\|_2 = 0$$

By theory of Riemann integral, $f = 0$ almost everywhere.

(b) We still have $\|f\|_2 = 0$. As f_1, f_2 cts

$$\Rightarrow f^2 \text{ cts } \geq 0 \quad \Rightarrow \quad f^2 \equiv 0. \quad \divideontimes$$

Cor 1.18 (Parseval's Identity)

For every 2π-periodic function f integrable on $[-\pi,\pi]$

$$\|f\|_2^2 = 2\pi a_0^2 + \pi\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

where a_0, a_n, b_n are Fourier coefficients of f.

Pf: By def. of a_n:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\,dx \Rightarrow \sqrt{2\pi}\,a_0 = \langle f, \tfrac{1}{\sqrt{2\pi}}\rangle_2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \Rightarrow \sqrt{\pi}\,a_n = \langle f, \tfrac{1}{\sqrt{\pi}}\cos nx\rangle_2$$

$n \geq 1$.

Similarly $\sqrt{\pi}\,b_n = \langle f, \frac{1}{\sqrt{\pi}}\sin nx\rangle_2$, $n\geq 1$.

Then $\langle f, S_N f\rangle_2 = \langle \underbrace{(f-S_N f)}_{\text{orthogonal to the subsp}} + S_N f, S_N f\rangle_2$

(by Cor 1.15)

$$= \langle S_N f, S_N f\rangle_2$$

$$= \int_{-\pi}^{\pi}\Big(a_0 + \sum_{k=1}^{N} a_k\cos kx + b_k \sin kx\Big)^2 dx$$

$$= 2\pi a_0^2 + \sum_{k=1}^{N}(\pi a_k^2 + \pi b_k^2)$$

Hence

$$0 \overset{\text{Thm 1.16}}{=} \lim_{N\to\infty} \|f - S_N f\|_2^2$$

$$= \lim_{N\to\infty} \left(\|f\|_2^2 - 2\langle f, S_N f\rangle_2 + \|S_N f\|_2^2 \right)$$

$$= \lim_{N\to\infty} \left(\|f\|_2^2 - 2\|S_N f\|_2^2 + \|S_N f\|_2^2 \right)$$

$$= \lim_{N\to\infty} \left(\|f\|_2^2 - \|S_N f\|_2^2 \right)$$

$$\therefore \|f\|_2^2 = \lim_{N\to\infty} \left[2\pi a_0^2 + \pi \sum_{k=1}^{N} (a_k^2 + b_k^2) \right]$$

\#

eg: By Fourier series of $f_1(x) = x$ on $[-\pi, \pi]$ and Parserval's Identity

$$(\text{HW2, Q5}) \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad (\text{Euler famula})$$

[The applications to the Wirtinger's Inequality and the Isoperimetric Problem (Cor 1.19 & §1.6 of my notes of the year 2016/17) will be omitted since they were removed from Prof Chou's notes in the last couple of years already.]

Ch2 Metric Space

In this chapter, X always denotes a non-empty set.

Def: A metric on X is a function

$$d : X \times X \to [0, +\infty) \text{ such that}$$

$\forall\, x, y, z \in X$

(M1) $d(x,y) \geq 0$ & "equality holds $\Leftrightarrow x = y$".

(M2) $d(x,y) = d(y,x)$

(M3) $d(x,y) \leq d(x,z) + d(z,y)$

The pair (X, d) is called a metric space.

Note: Condition (M3) is called the triangle inequality.

Def: Let (X, d) be a metric space. The metric ball of radius r centered at x

or simple the ball $B_r(x) = \{y \in X : d(y,x) < r\}$

eg 2.1 $(X = \mathbb{R},\ d(x,y) = |x-y|)$ is a metric space.

eg 2.2 Let $X = \mathbb{R}^n$, $d_2(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$
(Euclidean metric)

for $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n) \in \mathbb{R}^2$.

Then $(\mathbb{R}^n, d_2)$ is a metric space.

Recall the proof: $\|x\|^2 = \sum_{i=1}^{n} x_i^2$

Then $\|x+y\|^2 = \langle x+y, x+y \rangle = \|x\|^2 + 2\langle x,y\rangle + \|y\|^2$

By Cauchy-Schwarz inequality

$$|\langle x, y\rangle| \leq \|x\| \|y\|$$

$$\Rightarrow \quad \|x+y\|^2 \leq (\|x\| + \|y\|)^2$$

$$\Rightarrow \quad \|x+y\| \leq \|x\| + \|y\|, \quad \forall x, y \in \mathbb{R}^n$$

Replace x by $x-z$
y by $z-y$,

then $$\|x-y\| \leq \|x-z\| + \|z-y\|.$$

eg 2.3 Let $X = \mathbb{R}^n$, $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$

$$d_\infty(x,y) = \max_{i=1,\dots,n} |x_i - y_i|$$

Then $(\mathbb{R}^n, d_1)$ & $(\mathbb{R}^n, d_\infty)$ are metric spaces.

Generalization of egs 2.2 & 2.3 to function space:

eg 2.4 Let $C[a,b] = \{$ (real) continuous functions on $[a,b]\}$

$\forall f, g \in C[a,b]$, define

$$d_\infty(f,g) = \|f-g\|_\infty = \max\{|f(x)-g(x)| : x \in [a,b]\}$$

Then $(C[a,b], d_\infty)$ is a metric space (Ex!)

Similary, one can define

$$d_1(f,g) = \int_a^b |f(x)-g(x)|\, dx.$$

It is also easy to verify that $(C[a,b], d_1)$ is a metric space.

The natural generalization of the Euclidean metric to $C[a,b]$ is

$$d_2(f,g) = \sqrt{\int_a^b |f-g|^2}.$$

(M1) & (M2) are clear for d_2 (as f, g cts)

To see (M3), note that $d_2(f,g) = \|f-g\|_2$

Question 3 in HW 2 $\Rightarrow$ d_2 satisfies (M3).

$\therefore$ $(C[a,b], d_2)$ is a metric space.

<u>eg 25</u> On $X = R[a,b] = \left\{ \begin{array}{l} \text{Riemann integrable functions} \\ \text{on } [a,b] \end{array} \right\}$

d_1 is still defined $d_1(f,g) = \int_a^b |f-g|$

However, (M1) doesn't satisfied as

$$d_1(f,g) = 0 \Leftrightarrow f = g \text{ almost everywhere}$$
$$\not\Rightarrow f = g.$$

$\therefore$ d_1 is <u>not</u> a metric on $R[a,b]$.

To overcome this, we consider $X = R[a,b]/\sim$

where "$\sim$" is an <u>equivalent</u> relation on $R[a,b]$

defined by $\quad f\sim g \iff f=g$ almost everywhere.

(Check: "$\sim$" is an equivalent relation.)

Then elements of $R[a,b]/\sim$ can be represented as

$$[f] \text{ or } \bar{f} = \{\, g\in R[a,b] : g=f \text{ almost everywhere} \,\}$$

Now define $\tilde{d}_1$ on $R[a,b]/\sim$ by

$$\tilde{d}_1(\bar{f},\bar{g}) = d_1(f,g)$$

<u>Check</u>: $\tilde{d}_1$ is well-defined, i.e. indep. of the choice of representitivites f & g:

$\forall\, f_1\in\bar{f},\ g_1\in\bar{g}$. Then

$$d_1(f_1,g_1) = \int|f_1-g_1| \le \cancelto{0}{\int|f_1-f|} + \int|f-g| + \cancelto{0}{\int|g-g_1|}$$
$$= d_1(f,g)$$

Similarly $\quad d_1(f,g) \le d_1(f_1,g_1)$

$$\therefore\quad d_1(f,g) = d_1(f_1,g_1).$$

Then it is straigh forward to verify that

$(R[a,b]/\sim, \hat{d}_1)$ is a metric space.

Similarly for $(R[a,b]/\sim, \hat{d}_2)$ is a metric space

& note that $\tilde{d}_2$ is the L^2-distance we defined before.

Def: A <u>norm</u> $\|\cdot\|$ is a function on a real vector space X to $[0,+\infty)$ s.t. $\forall x, y \in X$, & $\alpha \in \mathbb{R}$,

(N1) $\|x\| \geq 0$ & "$\|x\| = 0 \Leftrightarrow x = 0$"

(N2) $\|\alpha x\| = |\alpha| \, \|x\|$

(N3) $\|x+y\| \leq \|x\| + \|y\|$

The pair $(X, \|\cdot\|)$ is called a <u>normed space</u>.

And $d(x,y) \overset{\text{def}}{=\!=} \|x-y\|$ is called the <u>metric induced</u> by the norm $\|\cdot\|$.

(Ex: Show that $d(x,y) = \|x-y\|$ is a metric with the property $d(\alpha x, \alpha y) = |\alpha| \, d(x,y)$, $\forall \alpha \in \mathbb{R}$)

egs: $\|X\|_2 = (\sum x_i^2)^{1/2}$, $\|X\|_1 = \sum |x_i|$,

$\|X\|_\infty = \max\{|x_1|, \dots, |x_n|\}$

are norms on $\mathbb{R}^n$

$\|f\|_2 = \left(\int_a^b |f|^2\right)^{1/2}$, $\|f\|_1 = \int_a^b |f|$,

$\|f\|_\infty = \max\{|f(x)| : x \in [a,b]\}$

are norms on $C[a,b]$.

We've seen "norm" $\xrightarrow{\text{induces}}$ "metric"

But not all metric induced from norm.

eg: X = non-empty set.

$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$ discrete metric on X.

- X not necessary a vector space, so d not induced by norm.
- Even for vector space:
  $\left.\begin{matrix}1\\0\end{matrix}\right\} = d(\alpha x, \alpha y) = |\alpha|\, d(x,y) = \begin{cases}|\alpha| \\ 0\end{cases}$
  Contradiction for $|\alpha| \neq 1$. (for $x \neq y$).

Def: Let (X, d) be a metric space.
Then for any non-empty $Y \subset X$,
$(Y, d|_{Y \times Y})$ is called a metric subspace of (X, d)

Notes: (i) metric subspace is a metric space.
(ii) We simple write (Y, d) for $(Y, d|_{Y \times Y})$.
(iii) A metric subspace of a normed space needs not be a normed space, only if the subset is also a vector subspace.

Def: A sequence $\{x_n\}$ in a metric space (X, d) is said to be converge to $x \in X$ if

$$\lim_{n \to \infty} d(x_n, x) = 0.$$

In this case, we write $\lim_{n \to \infty} x_n = x$, or $x_n \to x$ in X.

Prop (Uniqueness of limit)

If $x_n \to x$ & $x_n \to y$ in a metric space, then $x = y$.

(Pf: Same as $\mathbb{R}^n$)

egs: (i) Convergence in $(\mathbb{R}^n, d_2)$ is the usual convergence in adv. calculus.

(ii) Convergence in $(C[a,b], d_\infty)$ is the uniform convergence of seq. of functions in $C[a,b]$.