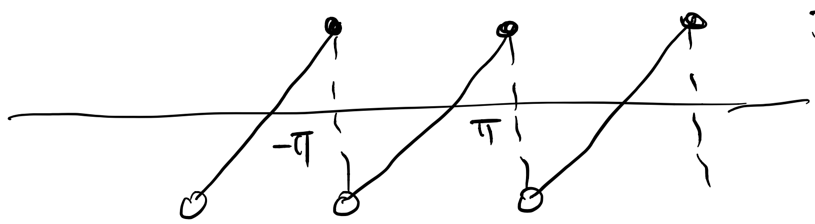
Thm 1.5 Let f be a 2π-periodic function integrable on $[-\pi,\pi]$. Suppose that f is Lipschitz continuous at x. Then $\{S_n f(x)\}$ converges to $f(x)$ as $n \to +\infty$.

(Pf: later at the end of this section)

eg of application:

Recall $f_1(x) = x$ on $[-\pi, \pi]$

$\hat{f}_1$ 2π-periodic extension



Fourier series

$$x \sim 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$$

It is clear that $f_1(x)$ is Lip. cts at any $x \in (-\pi, \pi)$

$$\therefore \quad \lim_{N\to+\infty} 2\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} \sin(nx) = x \quad \forall\, x \in (-\pi, \pi).$$

On the other hand, $\hat{f}_1$ is discts. at $x = \pm\pi$,

and we've have seen $\quad \hat{f}_1(\pm\pi) \neq 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n(\pm\pi).$

Thm 1.6 Let f be a 2π-periodic function integrable on $[-\pi, \pi]$.

Suppose that for $x_0 \in [-\pi, \pi]$

(i) $f(x_0^+) = \lim_{x \to x_0^+} f(x)$, $f(x_0^-) = \lim_{x \to x_0^-} f(x)$ both exist.

(right-hand limit) (left-hand limit)

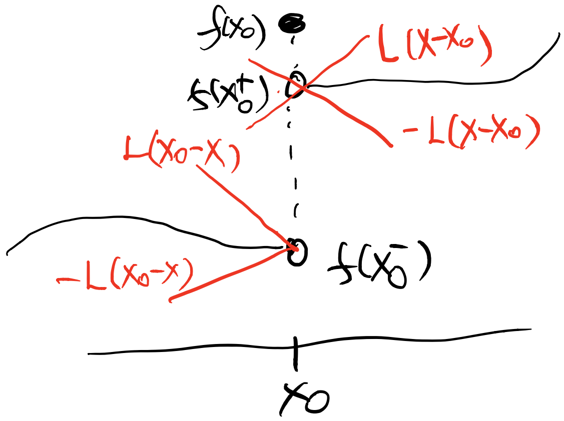
(ii) $\exists\, L > 0$ and $\delta > 0$ such that

$$|f(x) - f(x_0^+)| \le L(x - x_0), \quad 0 < x - x_0 < \delta$$

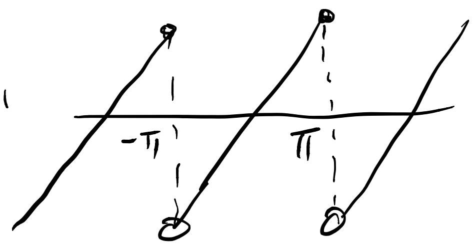
& $$|f(x) - f(x_0^-)| \le L(x_0 - x), \quad 0 < x_0 - x < \delta.$$

Then $$S_n f(x_0) \longrightarrow \frac{f(x_0^+) + f(x_0^-)}{2} \quad \text{as } n \to +\infty.$$

(Pf = Omitted)



eg of application : $f_1(x) = x$ with $\tilde{f}_1$



At $x_0 = \pi$, $\tilde{f}_1$ is discontinuous

(i) $f(\pi^+) = \lim_{x \to \pi^+} \tilde{f}_1(x) = -\pi$

$f(\pi^-) = \lim_{x \to \pi^-} \tilde{f}_1(x) = \pi$

(ii) For $0 < x - x_0 < \frac{\pi}{2}$ (ie $0 < x - \pi < \frac{\pi}{2}$)

$(\delta = \frac{\pi}{2})$

we have $|f(x) - f(\pi^+)|$

$= |f(x-2\pi) - (-\pi)|$ $\quad (x-2\pi \in (-\pi, \pi))$

$= |x - 2\pi - (-\pi)|$

$= x - \pi \leq L(x-\pi)$ with $L = 1$.

Similar for $0 < x_0 - x < \frac{\pi}{2}$.

Hence conditions of Thm 1.6 are satisfied

$\Rightarrow$ Fourier series $\underset{0}{\overset{\parallel}{S_n f(\pi)}} \to \frac{f(\pi^+) + f(\pi^-)}{2} = \frac{-\pi + \pi}{2}$

$= 0$.

Next we turn to "uniform" convergence and need

Def: A function f defined on $[a,b]$ is called to satisfy a Lipschitz condition if $\exists L>0$ such that

$$|f(x)-f(y)| \le L|x-y|, \quad \forall x,y \in [a,b].$$

Notes: (1) $L>0$ is indep. of $x,y \in [a,b]$

a kind of "uniform" Lip condition.

(2) f satisfies a Lip. condition $\Rightarrow$ f is Lip cts at every point on $[a,b]$.

eg: If $f \in C^1[a,b]$, $\Rightarrow$ $|f(y)-f(x)| = \left|\int_x^y f'(t)\,dt\right|$

$$\le M|y-x|, \quad \forall x,y \in [a,b]$$

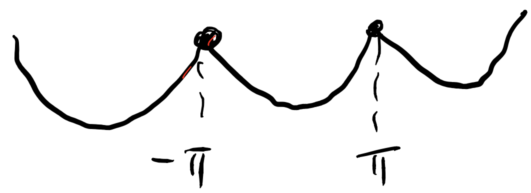
where $M = \sup_{[a,b]} |f'|$.

But $f(x)=|x|$ satisfies a Lip. condition, but not C^1.

Thm 1.7 Let f be a 2π-periodic function satisfying a Lipschitz condition. Then its Fourier series converges <u>uniformly</u> to f itself. (Pf: Omitted)

<u>eg of application</u> $f_2(x) = x^2$ on $[-\pi, \pi]$

$\tilde{f}_2$ 2π-periodic extension =



$\tilde{f}_2$ satisfies a Lip. condition (Check!)

$$\Rightarrow \quad \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx \quad \text{converges uniformly}$$

to x^2 on $[-\pi, \pi]$. ※

(Ex: put $x=0$ and get $\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.)

Proof of Thm 1.5

Let f be Lip cts at a point $x_0 \in [-\pi, \pi]$.

Step 1: $(S_n f)(x_0) = a_0 + \sum_{k=1}^{n} (a_k \cos kx_0 + b_k \sin kx_0)$

$$= \int_{-\pi}^{\pi} D_n(z) f(x_0+z)\,dz$$

where
$$D_n(z) = \begin{cases} \dfrac{\sin(n+\frac{1}{2})z}{2\pi \sin\frac{1}{2}z}, & \text{if } z \neq 0 \\ \dfrac{2n+1}{2\pi}, & \text{if } z = 0 \end{cases}$$

is called the Dirichlet kernel.

Pf: $(S_n f)(x_0) = a_0 + \sum_{k=1}^{n} (a_k \cos kx_0 + b_k \sin kx_0)$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\,dy + \sum_{k=1}^{n}\left[\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\cos ky\,dy\right)\cos kx_0 + \left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\sin ky\,dy\right)\sin kx_0\right]$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{1}{2} + \sum_{k=1}^{n}(\cos ky \cos kx_0 + \sin ky \sin kx_0)\right] f(y)\,dy$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{1}{2} + \sum_{k=1}^{n}\cos k(y-x_0)\right] f(y)\,dy$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{1}{2}+\sum_{k=1}^{n}\cos kz\right] f(z+x_0)\,dz \quad (z=y-x_0)$$

& 2π-periodic

Since $\frac{1}{2}+\sum_{k=1}^{n}\cos kz = \dfrac{\sin(n+\frac{1}{2})z}{2\sin\frac{1}{2}z}$ for $z\neq 0$,

(Ex: Calculate $e^{-in\theta}+\cdots+1+\cdots+e^{in\theta}$ using $1+z+\cdots+z^k=\dfrac{z^{k+1}-1}{z-1}$.)

$$(S_nf)(x_0) = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\sin(n+\frac{1}{2})z}{2\sin\frac{1}{2}z} f(x_0+z)\,dz$$

$$= \int_{-\pi}^{\pi} D_n(z) f(x_0+z)\,dz \quad \#$$

<u>Step 2</u> (Properties of $D_n(z)$)

(1) $\int_{-\pi}^{\pi} D_n(z)\,dz = 1$

(2) $D_n(z)$ is even, cts, 2π-periodic on $[-\pi,\pi]$,

& $D_n\left(\frac{2k\pi}{2n+1}\right)=0$ for $k=-n,\cdots,0,\cdots,n$

(3) $\max\limits_{[-\pi,\pi]} D_n(z) = D_n(0) = \dfrac{2n+1}{2\pi}$

(4) $\forall \delta>0$, $\int_0^{\delta} |D_n(z)|\,dz \to +\infty$ as $n\to+\infty$.
($\delta<\pi/2$)

Pf: (1) Easy: by integrating $\int_{-\pi}^{\pi}\left(\frac{1}{2}+\sum_{k=1}^{n}\cos kz\right)dz$.

(2) & (3) are easy exercise. (to be cont'd)