Ch 1 Fourier Series

Def: (1) Trigonometric series (三角級數) on $[-\pi, \pi]$ is a series of functions of the form

$$\sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)$$

(where $a_n, b_n \in \mathbb{R}$) $(b_0 = 0)$

$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

(2) If $b_n = 0$, $\forall n$, it is called a cosine series

" $a_n = 0$, $\forall n \geq 0$ " " " sine series

Easy facts

(1) If $\sum_{n=0}^{\infty} |a_n|, \sum_{n=0}^{\infty} |b_n| < \infty$,

then $\sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)$

is uniformly and absolutely convergent.

In particular, if $|a_n|, |b_n| \leq \frac{C}{n^s}$, $s > 1$ (for some $C > 0$),

then $\sum_{n=0}^{\infty} |a_n|, \sum_{n=0}^{\infty} |b_n| < \infty$, hence uniformly and absolutely convergent. (Pf: By M-test & $|\cos nx| \leq 1$, $|\sin nx| \leq 1$)

(2) $\phi(x) \stackrel{\text{def}}{=} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

is a continuous function on $[-\pi, \pi]$ provided $\sum |a_n| < \infty$, $\sum |b_n| < \infty$.

(3) $\phi(x)$ defined in (2) is 2π-periodic.

Pf $\phi(x+2\pi) = \lim_{n\to\infty} \sum_{k=0}^{n} (a_k \cos(k(x+2\pi)) + b_k \sin(k(x+2\pi)))$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} (a_k \cos kx + b_k \sin kx)$$

$$= \phi(x)$$

$\divideontimes$

Def: Let f be a 2π-periodic function on $\mathbb{R}$ which is Riemann integrable on $[-\pi, \pi]$. Then the Fourier Series (or Fourier expansion) of f is the trigonometric series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

with

$\boxed{\text{Fourier coefficients of } f}$

$$\begin{cases} a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)\,dy \\ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \cos ny\,dy \\ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \sin ny\,dy \end{cases} \quad (n \geq 1)$$

Notes (1) a_0 = average of f over $[-\pi,\pi]$

(2) Fourier series depends on the global information of f on $[-\pi,\pi]$.

(3) $f_1 \equiv f_2$ almost everywhere on $[-\pi,\pi]$

$\Rightarrow$ f_1 & f_2 have the same Fourier Series.

(4) Fourier series of f depends only on $f|_{(-\pi,\pi)}$, independent of the values of f on the end points.

Motivation of the definition of Fourier Series:

"If" $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$, $\forall x \in \mathbb{R}$

(& assume uniformly convergent.)

Then

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx$$

$$= a_0\int_{-\pi}^{\pi}\cos mx\,dx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos nx\cos mx\,dx + b_n\int_{-\pi}^{\pi}\sin nx\cos mx\,dx\right)$$

It is easy to calculate

- $$\int_{-\pi}^{\pi}\cos mx\,dx = \begin{cases} 2\pi & \text{if } m=0 \\ 0 & \text{if } m\neq 0 \end{cases}$$

- $$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \begin{cases} \pi, & \text{if } m=n \\ 0, & \text{if } m \neq n \end{cases}$$

- $$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0, \quad \forall n, m \geq 1$$

Hence if $m=0$,
$$\left.\begin{aligned} L.H.S. &= \int_{-\pi}^{\pi} f(x)\,dx \\ R.H.S. &= 2\pi a_0 \end{aligned}\right\} \Rightarrow \underline{a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx}$$

if $m>0$, then
$$\left.\begin{aligned} L.H.S. &= \int_{-\pi}^{\pi} f(x)\cos mx\,dx \\ R.H.S. &= a_m \cdot \pi \end{aligned}\right\} \Rightarrow \underline{a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx}$$

Similarly,
$$\int_{-\pi}^{\pi} f(x)\sin mx\,dx = a_0\int_{-\pi}^{\pi}\sin mx\,dx + \sum_{n=1}^{\infty}\left(a_n \overbrace{\cancel{\int_{-\pi}^{\pi}\cos nx \sin mx\,dx}}^{0} + b_n \int_{-\pi}^{\pi}\sin nx \sin mx\,dx \right)$$

Using
$$\int_{-\pi}^{\pi}\sin mx\,dx = 0, \quad \forall m$$
$$\int_{-\pi}^{\pi}\sin nx \sin mx\,dx = \begin{cases} \pi & \text{if } m=n \\ 0 & \text{if } m \neq n \end{cases}$$

Hence
$$\underline{b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx}.$$