The Chinese University of Hong Kong
2020 - 2021 MATH 2230A
Tutorial 1: Basic Structures of Complex Numbers
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Definition 0.1 (Complex Numbers)**.** The set of *complex numbers* $\mathbb{C}$ is the extension field of the set of real numbers $\mathbb{R}$ such that the polynomial x^2+1 has roots. If we denote i a root of of x^2+1, then we can say $\mathbb{C} := \{x+iy | x, y \in \mathbb{R}\}$.

Remark. Please refer to MATH2070/MATH3040 for the precise definition of regarding $\mathbb{C}$ as a field extension of $\mathbb{R}$.

There are different ways to parametrize, or represent, a complex number using two real numbers.

Definition 0.2 (Planar Parametrization)**.** Define a function $F : \mathbb{R}^2 \to \mathbb{C}$ by $(x, y) \mapsto x + iy$. Then we call F the *planar parametrization* of complex numbers. If $z = F(x, y) = x + iy$ for $x, y \in \mathbb{R}$, we call its *real part* $\mathrm{Re}(z) := x$ and its *imaginary part* $\mathrm{Im}(z) := y$.

Definition 0.3 (Polar Parametrization)**.** Define a function $G : (0, \infty) \times (-\pi, \pi]$ by $(\rho, \theta) \mapsto \rho e^{i\theta}$. Then we call G the *polar, or exponential parametrization* of complex numbers. If $z = G(\rho, \theta) = \rho e^{i\theta}$, we call its *modulus* $|z| := \rho$ and its *(principal) argument* $\mathrm{Arg}(z) := \theta$.

Remark. We can in fact transit between between the planar parametrization and the polar parametrization of complex numbers using the defining *Euler formula*, which says for all $\rho > 0$, $\theta \in \mathbb{R}$, we have the following

$$\rho e^{i\theta} := \rho \cos\theta + i\rho \sin\theta$$

Recall that the Euler formula is necessary if we want the exponential function to have "good" properties, namely the Taylor Series property and the identity $e^{x+y} = e^x e^y$ for $x, y \in \mathbb{C}$ (see Lecture 2).

1 Algebraic Structures

By definition, the set of complex number forms a *field.* Equivalently, we have 1. addition and multiplication are commutative and associative; 2. inverses and identities for both operations exist; 3. multiplication is distributive over addition. In addition to these basic operations, the complex numbers have, over the real numbers, an additional operation, namely the complex conjugation.

Definition 1.1. Let $z \in \mathbb{C}$. Suppose $z := x + iy$ for $x, y \in \mathbb{R}$. We define $\overline{z} := x - iy$ the *complex conjugate* of z.

Complex conjugation, despite its simple definition, indeed satisfies lots of nice properties, which are easy to prove as well.

Proposition 1.2. *Let $z \in \mathbb{C}$. Then we have the following:*

1. $\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z})$
2. $\mathrm{Im}(z) = \frac{1}{2i}(z - \overline{z})$
3. $z \in \mathbb{R}$ *if and only if* $z = \overline{z}$

Remark. From 3, we can see the conjugate operation trivializes if we consider only real numbers.

Proposition 1.3. *Let $z_1, z_2 \in \mathbb{C}$. Then we have the following:*

1. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ *(compatible with addition)*
2. $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$ *(compatible with multiplication)*
3. $\overline{1} = 1$ *(compatible with identity)*
4. $\overline{\overline{z}} = z$ *(involutive)*

*These properties are the so-called *-properties (star-properties).*

Proposition 1.4. *Let $z \in \mathbb{C}$. Then $z \cdot \overline{z} \in \mathbb{R}$ and $z \cdot \overline{z} = |z|^2$.*

Remark. This is indeed a very useful and in fact deep property relating conjugation and modulus.

2 Distance Structures

In $\mathbb{R}$, we can measure the distance between two points x, y using the absolute value and consider $|x - y|$. In $\mathbb{C}$, we can do the same with modulus, thanks to the important Triangle Inequality, which makes the modulus act like a *distance*.

Theorem 2.1 (Triangle Inequality for C)**.** *Let $z_1, z_2 \in \mathbb{C}$. Then we have the following,*

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

In the Lecture Note, a geometric proof is given for 2.1. In this tutorial, an algebraic proof is given. In fact, we shall see that the triangle inequality is equivalent to the Cauchy-Schwarz inequality (for 2 pairs of numbers), which you have learnt in MATH1030. Let's recall it.

Theorem 2.2 (Cauchy-Schwarz Inequality)**.** *Let $x_i, y_i \geq 0$ be a finite list of non-negative real numbers. Then we have the following*

$$\sum_i x_i y_i \leq (\sum_i x_i^2)^{\frac{1}{2}} (\sum_i y_i^2)^{\frac{1}{2}}$$

Proof of 2.1 using 2.2. Let $z_1 = x_1 + ix_2$ and $z_2 = y_1 + iy_2$ where $x_1, x_2, y_1, y_2 \in \mathbb{R}$. It suffices to show

$$(\sum_{i=1,2} |x_i + y_i|^2)^{\frac{1}{2}} \leq (\sum_{i=1,2} |x_i|^2)^{\frac{1}{2}} + (\sum_{i=1,2} |y_i|^2)^{\frac{1}{2}}$$

This follows immediately from the following chain of inequalities

$$\begin{aligned}
& \sum_{i=1,2} |x_i + y_i|^2 \\
\text{(triangle inequality for } \mathbb{R}) \leq & \sum_{i=1,2} |x_i + y_i|(|x_i| + |y_i|) \\
= & \sum_{i=1,2} |x_i + y_i||x_i| + \sum_{i=1,2} |x_i + y_i||y_i| \\
\text{(Cauchy-Schwarz inequality)} \leq & (\sum_{i=1,2} |x_i + y_i|^2)^{\frac{1}{2}} (\sum_{i=1,2} |x_i|^2)^{\frac{1}{2}} + \leq (\sum_{i=1,2} |x_i + y_i|^2)^{\frac{1}{2}} (\sum_{i=1,2} |y_i|^2)^{\frac{1}{2}} \\
= & (\sum_{i=1,2} |x_i + y_i|^2)^{\frac{1}{2}} ((\sum_{i=1,2} |x_i|^2)^{\frac{1}{2}} + (\sum_{i=1,2} |y_i|^2)^{\frac{1}{2}})
\end{aligned}$$

□

3 Order Structures

As an extension of real numbers, the set of complex numbers seem to satisfy more properties. Nonetheless, complex numbers are less ideal in terms their order structures.

Definition 3.1. Let $\leq$ be a relation on $\mathbb{C}$. Then

1. we call $\leq$ *reflexive* if $x \leq x$ for all $x \in \mathbb{C}$
2. we call $\leq$ *transitive* if $x \leq y$, and $y \leq z$ imply $x \leq z$ for all $x, y, z \in \mathbb{C}$
3. we call $\leq$ *symmetric* if $x \leq y$ and $y \leq x$ imply $x = y$ for all $x, y \in \mathbb{C}$
4. we call $\leq$ *total* if $x \leq y$ or $y \leq x$ for all $x, y \in \mathbb{C}$
5. we call $\leq$ *compatible with addition* if $x \leq y$ implies $x + z \leq y + z$ for all $x, y, z \in \mathbb{C}$
6. we call $\leq$ *compatible with product* if $x \leq y$ implies $xz \leq yz$ for all $x, y \in \mathbb{C}$ and $0 \leq z$

We call $\leq$ a *preorder* if it is reflexive and transitive; we call a symmetric preorder a *partial ordering*; and we call a total partial ordering a *total ordering.*

When we say the set of complex numbers lack an order structure, we mean precisely the following.

Theorem 3.2. *There is no total ordering compatible with both addition and product for* $\mathbb{C}$.

Proof. We shall give a proof by contradiction. Let's first suppose there is one,denoted by $\leq$, so by totality either $0 \leq i$ or $i \leq 0$. Let's suppose $0 \leq i$. Then by product compatibility, we have $0 \leq i^2 = -1$. From this, we have $0 \leq 1$ by product compatibility again, or we have $1 \leq 0$ by adding 1 on both sides. Then by symmetry, $1 = 0$, which is false. The case for $i \leq 0$ is left to the reader. $\square$

Remark. Although $\mathbb{C}$ has no total ordering that is compatible with the algebraic operations, it indeed admits total orderings if we do not require such compatibility (see Exericse below).

4 Exericse

1. Let $z = -2 + i$ and $w = 3 + 4i$. Compute the following in planar form.

a) $z + \overline{w}$ b) $-\overline{z}w$ c) $\frac{z}{iw}$

2. Let $z, w \in \mathbb{C}$.

(i). Prove that $z\overline{z} = |z|^2$

(ii). Prove that $|zw| = |z||w|$

(iii). Prove that $|z + w|^2 = |z|^2 + |w|^2 + 2\operatorname{Re}(z\overline{w})$

(iv). Give a counter example to show that the square of a complex number may not be a real number.

3. Show that $|\operatorname{Re}(z)| \leq |z|$ and $|\operatorname{Im}(z)| \leq |z|$ for all $z \in \mathbb{C}$.

4. Let $0 \neq z \in \mathbb{C}$. Show that there exists $w \in \mathbb{C}$ with $|w| = 1$ such that w is a scalar multiple of z, that is, there exists $\lambda \in \mathbb{R}$ such that $\lambda w = z$. We call such w a *normalization* of z.

5. Prove the inverse triangle inequality using triangle equality:

$$||z| - |w|| \leq |z - w|$$

where $z, w \in \mathbb{C}$

6. Prove the Cauchy Schwarz inequality (for two pairs of numbers) using Triangle Inequality for complex numbers; this, together with the proof in this note, shows that Triangle Inequality for complex number is equivalent to the Cauchy-Schwarz inequality for two pairs of real numbers, that is:

$$\sum_i x_i y_i \leq (\sum_i x_i^2)^{\frac{1}{2}} (\sum_i y_i^2)^{\frac{1}{2}}$$

where x_1, x_2, y_1, y_2 are non-negative real numbers.

7. Continue the proof of Theorem 3.2, which says that, there is no total ordering on $\mathbb{C}$ that is compatible with both addition and product.

8. Define a relation $\leq$ on $\mathbb{C}$ by the following:

$$\text{If } \mathrm{Re}(z) \leq \mathrm{Re}(w), \text{ then } z \leq w \tag{1}$$

$$\text{If } \mathrm{Re}(z) > \mathrm{Re}(w), \text{ then } z \leq w \text{ if } \mathrm{Im}(z) \leq \mathrm{Im}(w) \tag{2}$$

(i). Show that $\leq$ is a total ordering.

(ii). Is it true that $\leq$ is compatible with addition? If not, give a counter example.

(iii). Is it true that $\leq$ is compatible with product? If not, give a counter example.

This ordering is called the *lexicographic order*