MATH2230A Tutorial 1

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Your TA

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Please feel free to contact us if you have any question.

Table of Contents

2,3,4xxx courses VS 1xxx, secondary
~> Abstract spaces VS concrete numbers

First step: concrete representation; relate the abstract structure to what you know

Abstract Structures:
1030: Vector space
2070: group, ring, field
3060: metric space, etc.

Next Lecture/Lecture this week
After representations, and structures, probably subsets and functions.

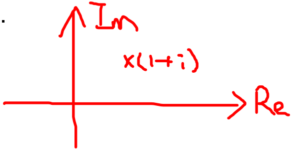
Planar Representation of Complex Numbers

Definition

The space of complex number $\mathbb{C}$ is the (field) extension of the space of real numbers $\mathbb{R}$ such that the real polynomial $x^2 + 1$ has a root. We denote the roots $i, -i$.

Definition (Planar parametrization)

Define a function $F : \mathbb{R}^2 \to \mathbb{C}$ by $(x, y) \mapsto x + iy$. Then we call F the *planar parametrization* of complex numbers. If $z = F(x, y) = x + iy$ for $x, y \in \mathbb{R}$, we call its *real part* $\mathrm{Re}(z) := x$ and its *imaginary part* $\mathrm{Im}(z) := y$.



Polar Representation and the Euler Forumula

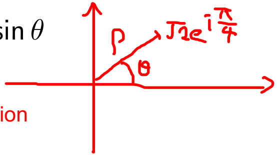
Definition (Polar Representation)

Define a function $G : (0, \infty) \times (-\pi, \pi]$ by $(\rho, \theta) \mapsto \rho e^{i\theta}$. Then we call G the *polar, or exponential parametrization* of complex numbers. If $z = G(\rho, \theta) = \rho e^{i\theta}$, we call its *modulus* $|z| := \rho$ and its *(principal) argument* $\mathrm{Arg}(z) := \theta$

Transition between Planar and Polar Form - Euler Forumla

For all $\rho > 0$, $\theta \in \mathbb{R}$, we have the following

$$\rho e^{i\theta} := \rho \cos\theta + i\rho \sin\theta$$



Remark: The Euler Formula is a definition.
It is necessary so that exp has the Taylor Series Expansion
and satisfies e^(x+y) = e^x e^y.

Table of Contents

Second Step:
Structures of the abstract space.
Algebraic, Distance and Order Structures are the basic structures we find in R.

The Field Properties

The Complex Numbers follows the following Field Properties and hence is a field.

1. $x + (y + z) = (x + y) + z$ and $(xy)z = x(yz)$ for all $x, y, z \in \mathbb{C}$. (Asso. of $+, \times$)
2. $x + y = y + x$ and $xy = yx$ for all $x, y \in \mathbb{C}$. (Comm. of $+, \times$)
3. There exists $0, 1$ such that $0 + x = x$ and $1x = x$ for all $x \in \mathbb{C}$. (Id. of $+, \times$)
4. For all $x \in \mathbb{C}$, there exists $y \in \mathbb{C}$ such that $x + y = 0$. (Inv. of $+$)
5. For all $0 \neq x \in \mathbb{C}$, there exists $y \in \mathbb{C}$ such that $xy = 1$. (Inv. of $\times$)

What is the geometric meaning of $+$ and $\times$?

Addition is translate
Multiplication is rotation with possibly a re-scaling.

Complex Conjugation

Complex conjugate is reflection about the real axis.

Definition

Let $z \in \mathbb{C}$. Suppose $z := x + iy$ for $x, y \in \mathbb{R}$. We define $\overline{z} := x - iy$ the *complex conjugate* of z.

Proposition (Planar representation and Complex conjugate)

Let $z \in \mathbb{C}$. Then we have the following:

a) $\operatorname{Re}(z) = \frac{1}{2}(z + \overline{z})$ *b)* $\operatorname{Im}(z) = \frac{1}{2i}(z - \overline{z})$

c) *$z \in \mathbb{R}$ if and only if $z = \overline{z}$* (conjugate is trivial in real numbers)

Proposition (The star properties)

(Complex Conjugate works well with add. and mul.)

Let $z_1, z_2 \in \mathbb{C}$. Then we have the following star properties:

1. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ *(compatible with addition)*
2. $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$ *(compatible with multiplication)*
3. $\overline{1} = 1$ *(compatible with identity)*
4. $\overline{\overline{z}} = z$ *(involutive)*

Complex Conjugate

Alg: + ⟼ −

Geo length

Proposition (Conjugate - Modulus Formula)

Let $z \in \mathbb{C}$. Then $z\bar{z} = |z|^2$.

You have 3 minutes to prove all these properties.

Question:

Let $z, w \in \mathbb{C}$. Prove that $|zw| = |z||w|$

Proof using the Conjugate - Modulus Formula

This is equivalent to proving $|zw|^2 = |z|^2|w|^2$.
By the Conjugate modulus formula, we have the following

$$L.H.S = |zw|^2 = zw\overline{zw} = zw\bar{z}\overline{w} = z\bar{z}w\overline{w} = |z|^2|w|^2 = R.H.S$$

□

Table of Contents

Distance => Nearness between two points => convergence, limit, differentiation and so on. (topology)

Basic Inequalities

Theorem (Triangle Inequality for $\mathbb{C}$)

Let $z_1, z_2 \in \mathbb{C}$. Then we have the following,

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

In fact, The Triangle Inequality for $\mathbb{C}$ is equivalent to the Cauchy-Schwarz Inequality for two pairs of real numbers.

Theorem (Cauchy-Schwarz Inequality)

Let $x_i, y_i \geq 0$ be a finite list of non-negative real numbers. Then we have the following

$$\sum_i x_i y_i \leq \left(\sum_i x_i^2\right)^{\frac{1}{2}} \left(\sum_i y_i^2\right)^{\frac{1}{2}}$$

Math 1030: $|\langle x, y\rangle| \leq \|x\|_2 \|y\|_2$

An Alternate Proof of Triangle Inequality

Proof of Triangle Inequality using Cauchy-Schwarz Inequality.

Let $z_1 = x_1 + ix_2$ and $z_2 = y_1 + iy_2$ where $x_1, x_2, y_1, y_2 \in \mathbb{R}$. The Triangle Inequality can be rewritten as

$$\Big(\sum_{i=1,2} |x_i + y_i|^2\Big)^{\frac{1}{2}} \le \Big(\sum_{i=1,2} |x_i|^2\Big)^{\frac{1}{2}} + \Big(\sum_{i=1,2} |y_i|^2\Big)^{\frac{1}{2}}$$

This follows immediately from the following chain of inequalities

(Triangle inequality for real numbers)

$$\sum_{i=1,2} |x_i + y_i|^2 \le \sum_{i=1,2} |x_i + y_i|(|x_i| + |y_i|)$$

$$= \sum_{i=1,2} |x_i + y_i||x_i| + \sum_{i=1,2} |x_i + y_i||y_i|$$

(Cauchy-Schwarz Inequalities)

$$\le \Big(\sum_{i=1,2} |x_i + y_i|^2\Big)^{\frac{1}{2}}\Big(\Big(\sum_{i=1,2} |x_i|^2\Big)^{\frac{1}{2}} + \Big(\sum_{i=1,2} |y_i|^2\Big)^{\frac{1}{2}}\Big)$$

□

Table of Contents

Basic definitions

Let $\leq$ be a relation on $\mathbb{C}$. Then (math 1050)

1. we call $\leq$ *reflexive* if $x \leq x$ for all $x \in \mathbb{C}$
2. we call $\leq$ *transitive* if $x \leq y$, and $y \leq z$ imply $x \leq z$ for all $x, y, z \in \mathbb{C}$
3. we call $\leq$ *symmetric* if $x \leq y$ and $y \leq x$ imply $x = y$ for all $x, y \in \mathbb{C}$
4. we call $\leq$ *total* if $x \leq y$ or $y \leq x$ for all $x, y \in \mathbb{C}$
5. we call $\leq$ *compatible with addition* if $x \leq y$ implies $x + z \leq y + z$ for all $x, y, z \in \mathbb{C}$
6. we call $\leq$ *compatible with product* if $x \leq y$ implies $xz \leq yz$ for all $x, y \in \mathbb{C}$ and $0 \leq z$

We call $\leq$ a *preorder* if it is reflexive and transitive; we call a symmetric preorder a *partial ordering*; and we call a total partial ordering a *total ordering*.

The Lack of Good Order Structure

Theorem

There is no total ordering compatible with both addition and product for $\mathbb{C}$.

Remark: There are in fact total orderings on C if we do not require their compatibility with both addition and product (see Exercise in Tutorial Note).

Proof.

We shall give a proof by contradiction. Suppose there is one,denoted by $\leq$.
By totality either $0 \leq i$ or $i \leq 0$. Let's suppose $0 \leq i$.
By product compatibility, we have $0 \leq i^2 = -1$.
From this, we have $0 \leq 1$ by product compatibility again, or we have $1 \leq 0$ by adding 1 on both sides.
Then by symmetry, $1 = 0$, which is false.
Similar arguments exist for $i \leq 0$.
By contradiction, we must have that such a total ordering cannot exist in the first place. □

HW: Due on Friday.
One PDF file onto blackboard.
You are recommended to type your HW (e.g. by LaTeX). Overleaf is a good website to let you type LaTeX without downloading anything on your PC.
If you can't type, that's ok. But you have to make the HW is visible.
You are absolutely welcome to phototake your HW as long as the image is clear, but you should regard that as your last resort.

Thank you!