MATH 2230A - Home Test 1 - Solutions

Suggested Solutions *(It does not reflect the marking scheme)*

1. Discuss if the following two functions are analytic on $\mathbb{C}$ or not. Justify your conseqeunces with proofs. If $f(z)$ or $g(z)$ is analytic on $\mathbb{C}$, please compute its complex derivatives.

$$f(z) = e^{-y}\sin x - ie^{-y}\cos x \qquad\qquad g(z) = 2xy + i(x^2 - y^2)$$

Solution. $f(z)$: Note that $f(z) = e^{-y}\sin x - ie^{-y}\cos x$. Hence, $u(x,y) = e^{-y}\sin x$ and $v(x,y) = -e^{-y}\cos x$. We then have

$$\begin{aligned} u_x(x,y) &= e^{-y}\cos x & \qquad v_x(x,y) &= e^{-y}\sin x \\ u_y(x,y) &= -e^{-y}\sin x & \qquad v_y(x,y) &= e^{-y}\cos x \end{aligned}$$

for all $z \in \mathbb{C}$. Therefore the partial derivatives exist everywhere. Since the partial derivatives are compositions of exponential functions and trigonometric functions, they are continuous everywhere. Lastly, the Cauchy-Riemann equations are satisfied everywhere: $u_x(z) = v_y(z)$ and $u_y(z) = -v_x(z)$ for all $z \in \mathbb{C}$. By the sufficiency of Cauchy-Riemann Equations, f is analytic everywhere on $\mathbb{C}$.
We then proceed the complex derivative of f, which is given by

$$f'(z) = u_x(z) + iv_x(z) = e^{-y}\cos x + i\sin xe^{-y} = e^{-y}(\cos x + i\sin x) = e^{-y}e^{ix} = e^{i(x+iy)} = e^{iz}$$

$g(z)$: Note that $g(z) = 2xy + i(x^2 - y^2)$. Hence, $u(x,y) = 2xy$ and $v(x,y) = x^2 - y^2$. Then for all $z \in \mathbb{C}$, we have

$$\begin{aligned} u_x(x,y) &= 2y & \qquad v_x(x,y) &= 2x \\ u_y(x,y) &= 2x & \qquad v_y(x.y) &= -2y \end{aligned}$$

Note that we have

$$\begin{cases} u_x(z) = v_y(z) \\ u_y(z) = -v_x(z) \end{cases} \Leftrightarrow \begin{cases} 2y = -2y \\ 2x = -2x \end{cases} \Leftrightarrow \begin{cases} y = 0 \\ x = 0 \end{cases} \Leftrightarrow z = 0$$

So, the C-R equations are satisfied if and only if $z = 0$.
When $z = 0$, since all partial derivatives exist everywhere, they exist in a neighborhood of $z = 0$ in particular. Moreover, the C-R equations are satisfied at $z = 0$ and the derivatives are continuous at $z = 0$. Therefore, g is complex differentiable at 0.
When $z \neq 0$, the $C - R$ equations are not satisfied. So, g is not complex differentiable at $z \neq 0$.
To conclude, g is not analytic anywhere on $\mathbb{C}$ (since it is not complex differentiable everywhere on any open set and so any neighborhood of points).
(Since g is not analytic on $\mathbb{C}$, we don't have to compute its derivative).

2. Suppose that $f(z)$ is the power function z^{1+i} defined in the principal branch. C is the semi-circle

$$z = e^{i\theta}, \pi \leq \theta \leq 2\pi$$

Compute

$$\int_C f(z)dz$$

Solution. Let $\gamma : [\pi, 2\pi] \to \mathbb{C}$ by $\gamma(\theta) = e^{i\theta}$ be the parametrization of the semicircle in question. Since we are using the principal branch ($-\pi < \theta \leq \pi$), we have for all $\theta \in (\pi, 2\pi]$ that $\arg(e^{i\theta}) = \theta - 2\pi$. Then we compute that

$$\begin{aligned}
\int_C f(z)dz &= \int_\pi^{2\pi} f \circ \gamma(\theta)\gamma'(\theta)d\theta \\
&= \int_\pi^{2\pi} f(e^{i\theta})ie^{i\theta}d\theta \\
&= \int_\pi^{2\pi} e^{(1+i)\log\left(e^{i\theta}\right)}ie^{i\theta}d\theta \\
&= \int_\pi^{2\pi} e^{(1+i)i\arg\left(e^{i\theta}\right)}ie^{i\theta}d\theta \\
&= \lim_{\epsilon\to\pi^+} \int_\epsilon^{2\pi} e^{(1+i)i(\theta-2\pi)}ie^{i\theta}d\theta \\
&= \int_\pi^{2\pi} e^{(1+i)i(\theta-2\pi)}ie^{i\theta}d\theta \\
&= \int_\pi^{2\pi} ie^{2\pi}e^{(2i-1)\theta}d\theta \\
&= \frac{ie^{2\pi}}{2i-1} e^{(2i-1)\theta}\Big]_\pi^{2\pi} = \frac{i}{2i-1}(1-e^\pi) = \frac{1}{2+i}(1-e^\pi)
\end{aligned}$$

(Please note that we use impropert integral on the 5th line since the integrand does not match at the endpoint $z = \pi$. We can take out the limit afterwards because the integrad is continuous everywhere and hence the respective indefinite integral is continuous at the end-point.)

Remark: A very large portion of marks would be deducted if you did not care about the value of the argument function (and hence computed the wrong answer).

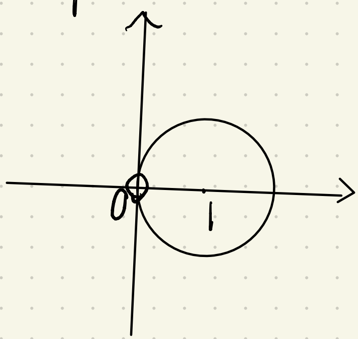
3. There are three questions in this problem. In each question, find out the corresponding complex numbers z which satisfy the relation given in the question:

a) $\mathrm{Re}(\frac{1}{z}) = \frac{1}{2}$ b) $e^z = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ c) $z = |(-i)^i|$

3. i) $\mathrm{Re}\left(\frac{1}{z}\right) = \frac{1}{2}$

$$\Rightarrow \begin{cases} \mathrm{Re}\left(\frac{1}{x+yi}\right) = \frac{1}{2} \\ z \neq 0 \ (x \neq 0, y \neq 0) \end{cases} \Rightarrow \begin{cases} \mathrm{Re}\left(\frac{x-yi}{x^2+y^2}\right) = \frac{1}{2} \\ x \neq 0,\ y \neq 0 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{x}{x^2+y^2} = \frac{1}{2} \\ x \neq 0,\ y \neq 0 \end{cases} \Rightarrow \begin{cases} x^2+y^2-2x = 0 \\ x \neq 0,\ y \neq 0 \end{cases} \Rightarrow \begin{cases} (x-1)^2+y^2=1 \\ x \neq 0,\ y \neq 0 \end{cases}$$



circle $|z-1| = 1$
$z \neq 0$

ii) $e^z = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$

$$\Rightarrow z = \log\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right) = \log\left(e^{\left(\frac{3}{4}\pi i + 2\pi n i\right)}\right)$$

$$= \frac{3}{4}\pi i + 2\pi n i, \quad n \in \mathbb{Z}$$

iii) $z = |(-i)^i| = |e^{i\log(-i)}|$

$$= |e^{i\left(-\frac{\pi}{2} + 2n\pi\right)i}| = e^{\frac{\pi}{2} - 2n\pi}, \quad n \in \mathbb{Z}.$$

4. Suppose that the power function $z^{1/2}$ is defined on the principal branch. Discuss the continuity of the function

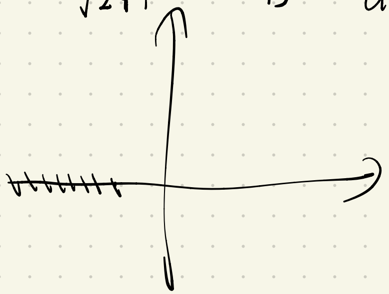
$$h(z) = (z^2 - 1)^{1/2}$$

at the origin 0. Please justify your consequences with proof.

4. No $\sqrt{z^2-1} = \sqrt{z+1}\;\sqrt{z-1}$

$\sqrt{z+1}$ is analytic on branch cut at 0,

$\Rightarrow \sqrt{z+1}$ cts at 0.

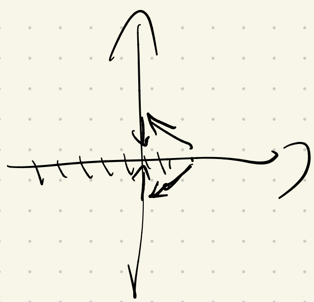


for $\sqrt{z-1}$, approach from $yi \to 0$, $(y>0)$

$\lim_{y^+ \to 0} \sqrt{yi-1} = \sqrt{1+y^2}\, e^{\frac{\pi}{2}i}$

approach from $yi \to 0$, $(y<0)$

$\lim_{y^- \to 0} \sqrt{yi-1} = \sqrt{1+y^2}\, e^{-\frac{\pi}{2}i}$



Not cts $\Rightarrow \sqrt{z^2-1}$ not cts at 0.

The reason why taking $0 \in \{z^2-1 \le 0\}$ as argument is wrong:

Any pt on imaginary axis except 0 is analytic, and $(iy)^2-1 = -y^2-1 < 0$ so it should not be analytic, but according to $\sqrt{z-1}\,\sqrt{z+1}$, it is analytic.