MATH 2230A - HW 8 - Solutions

Full solutions at P.219 - 220 Q4, 6, 7, 8

Commonly missed steps in Purple and common mistakes at the back

Through the solution, we use $B(x,r), \overline{B(x,r)}, C(x,r)$ to denote open balls, closed balls and circles (boundaries of balls) respectively.
We recall the following important facts for power series. In the following, we would call $f(z) := \sum_{n=0}^{\infty} a_n(z-z_0)^n$ where $a_n \in \mathbb{C}$ for all $n \in \mathbb{N}$ a *formal* power series centered at $z_0 \in \mathbb{C}$ (which does not necessarily converge).

Theorem 0.1 (Taylor's Theorem (as stated in the textbook))**.** *Let $z_0 \in \mathbb{C}$ and $r > 0$. Suppose f is analytic on the open ball $B(z_0, r)$, then its Taylor seires at z_0 converges pointwise to itself, that is, to f on $B(z_0, r)$. So we have for all $z \in B(z_0, r)$,*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

Remark. This is in fact equivalent to the version stated in the Lecture Note/ HW 7 solution. The proof was shown in the last solution in HW7.

Theorem 0.2 (Convergence of Power Series)**.** *Let $f(z)$ be a formal power series centered at z_0. Suppose $f(z)$ converges at some $w \neq z_0$. Then f converges on the open ball $B(z_0, R_\omega)$ where $R_\omega := |z-w|$. In fact $f(z)$ converges absolutely on $B(z_0, R_\omega)$*

Remark. This shows that if a power series converges on some points of the circle $C(z_0, r)$ then it converges on the open disk $B(z_0, r)$. That is why a power series always converges on some open disks (with some points on the boundary).

Theorem 0.3 (Uniform Convergence on Power Series)**.** *Let $f(z)$ be a formal power series centered at z_0 that converges on an open ball $B(z_0, r)$ with $r > 0$. Then for all $0 < \rho < r$, the power series converges uniformly on $\overline{B(z_0, \rho)}$.*

Remark. Please note that it does NOT follow that $f(z)$ converges uniformly on $B(z_0, r)$

Theorem 0.4 (Differentiation of power series)**.** *Let $f(z) := \sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a formal power series centered at z_0 and converges on $B(z_0, r)$ for $r > 0$. Then f is complex differentiable for all $z \in B(z_0, r)$ and we have*

$$D_z(f(w)) = f'(z) = \sum_{n=0}^{\infty} D_z(a_n(w-z_0)^n) = \sum_{n=1}^{\infty} a_n n(z-z_0)^{n-1}$$

where $D_z(g) := g'(z)$ for g analytic at z.

Remark. This is mainly because f converges uniformly on some neighborhood of z for any $z \in B(z_0, r)$.

Theorem 0.5 (Integration of Power Series)**.** *Let $f(z) := \sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a formal power series centered at z_0 and converges on $B(z_0, r)$ for $r > 0$. Then for any (continuously differentiable) contour C within the open ball, we have*

$$\int_C f(z)dz = \sum_{n=0}^{\infty} \int_C a_n(z-z_0)^n dz$$

Remark. This is mainly because we can enclose the contour C within some ball $B(z_0, \rho)$ where $\rho < r$ on which the power series converges uniformly. (This is possible as we consider contours C to be parametrized by compact (closed and bounded) intevals through some continuous $\gamma : [a, b] \to C$. One way to see why this is useful is to consider the map $t \mapsto |\gamma(t)|$. By the extreme value theorem, such ρ exists as a result. Then integral-sum exchange formula then follows by the uniform convergence of power series.

Solutions

P.219-220

2. By substituting $1/(1-z)$ for z in the expansion

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty}(n+1)\,z^n \qquad (|z| < 1),$$

found in Exercise 1, derive the Laurent series representation

$$\frac{1}{z^2} = \sum_{n=2}^{\infty} \frac{(-1)^n(n-1)}{(z-1)^n} \qquad (1 < |z-1| < \infty).$$

(Compare with Example 2, Sec. 71.)

Solution. It follows from a simple substitution. Please note that you have to verify why $|1/1-z| < 1$ before doing the substitution as the first identity is valid only for $|z| < 1$.

4. Show that the function defined by means of the equations

$$f(z) = \begin{cases} (1-\cos z)/z^2 & \text{when } z \neq 0, \\ 1/2 & \text{when } z = 0 \end{cases}$$

is entire. (See Example 1, Sec. 71.)

Solution. When $z \neq 0$, it is clear that f is holomorphic at z as f is quotients of elementary things there with non-zero denominator. It remains to show the analyticity of f at 0.
Note that $\cos z$ is entire. Hence by Taylor's Theorem, its Taylor series at 0 converges everywhere. In particular, we have $\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$ for all $z \in B(0,r)$ for some $r > 0$. Hence in a deleted neighborhood $B(0,r)\backslash\{0\}$ of 0, we have

$$f(z) = \frac{1-\cos z}{z^2} = \frac{1}{z^2}(1-(1-\frac{z^2}{2!}+\frac{z^4}{4!}+\ldots)) = \frac{1}{z^2}(\frac{z^2}{2!}-\frac{z^4}{4!}+\ldots) = \frac{1}{2}-\frac{z^2}{4!}+\frac{z^4}{6!}-\ldots$$

where the (formal) power series $h(z) = \frac{1}{2} - \frac{z^2}{4!} + \frac{z^4}{6!} - \ldots$ in fact converges at $z = 0$ with $h(0) = 1/2 = f(0)$. Therefore, $f(z) = h(z)$ on $B(0,r)$ and hence f is a power series on $B(0,r)$. By differentiability of power series, f is analytic at 0.

5. Prove that if

$$f(z) = \begin{cases} \dfrac{\cos z}{z^2-(\pi/2)^2} & \text{when } z \neq \pm\pi/2, \\ -\dfrac{1}{\pi} & \text{when } z = \pm\pi/2, \end{cases}$$

then f is an entire function.

Solution. It is clear that f is analytic if $z \neq \pm\pi/2$. You have to check analyticity at both points $\pm z/2$. The argument is similar to that of Question 4 and 7 except that in the last line you should claim that f coincides with a product of a power series with some other analytic functions (which is also analytic on the region concerned) instead of writing f coincides with a power series.

6. In the w plane, integrate the Taylor series expansion (see Example 1, Sec. 64)

$$\frac{1}{w} = \sum_{n=0}^{\infty}(-1)^n(w-1)^n \qquad (|w-1|<1)$$

along a contour interior to its circle of convergence from $w=1$ to $w=z$ to obtain the representation

$$\operatorname{Log} z = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^n \qquad (|z-1|<1).$$

Solution. Let $f(z) := 1/z$. Note that f is analytic everywhere except at $z=0$. In particular, $f(z)$ is analytic on the open ball $B(1,1)$. By Taylor's theorem, f is its Taylor series (centered at 1) on $B(1,1)$, with

$$f(w) = \frac{1}{w} = \sum_{n=0}^{\infty}(-1)^n(w-1)^n$$

for all $w \in B(1,1)$. Let $z \in B(1,1)$ and let C be a contour from 1 to z insides $B(1,1)$. Then by the integrability of power series, we have

$$\int_C \frac{1}{w}dw = \sum_{n=0}^{\infty}\int_C (-1)^n(w-1)^n dw$$

Note that now the functions $f(w) = 1/w$ have the anti-derivative $\operatorname{Log} w$ on the open, connected region $B(1,1)$ since $B(1,1)$ is away from the principal branch cut while $(w-1)^n, n \in \mathbb{N}$ clearly has antiderivatives on $B(1,1)$. Applying the fundamental theorem of contour integral, we have independence of paths for the above integrals and so we have

$$\begin{aligned}\operatorname{Log}(z) - \operatorname{Log}(1) = \int_C \frac{1}{w}dw &= \sum_{n=0}^{\infty}\int_C (-1)^n(w-1)^n dw \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n(w-1)^{n+1}}{n+1}\Bigg]_1^z = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(z-1)^n}{n}\end{aligned}$$

The result follows as $\operatorname{Log}(1) = 0$

7. Use the result in Exercise 6 to show that if

$$f(z) = \frac{\operatorname{Log} z}{z-1} \qquad \text{when } z \neq 1$$

and $f(1) = 1$, then f is analytic throughout the domain

$$0 < |z| < \infty, \ -\pi < \operatorname{Arg} z < \pi.$$

Solution. Let $\Omega := \{z||z| \in (0,\infty); \arg z \in (-\pi,\pi)\}$ be domain in question. Note that $\Omega = \mathbb{C}\backslash(-\infty,0]$, which is the complement of the principal branch cut.
When $z \neq 1$, by the definition of Ω, $\operatorname{Log}(z)$ is analytic on Ω (NOT on $\mathbb{C}\backslash\{1\}$) (why?) while it is clear that $z-1$ is non-zero analytic. Hence, f is analytic when $z \neq 1$. It remains to show the analyticity of f at $z=1$.
Note that $\operatorname{Log} z$ is analytic on $B(1,1)$, which is away from the branch cut. Hence by Taylor's Theorem, its Taylor series at 1 converges to itself on $B(1,1)$. By Q6, we have $\operatorname{Log} z = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(w-1)^n}{n}$ for all $z \in B(1,1)$. Hence in a deleted neighborhood $B(1,1)\backslash\{0\}$ of 1, we have

$$f(z) = \frac{\operatorname{Log} z}{z-1} = \frac{1}{z-1}((z-1) - \frac{1}{2}(z-1)^2 + \frac{1}{3}(z-1)^3 - \ldots) = 1 - \frac{1}{2}(z-1) + \frac{1}{3}(z-1)^2 - \ldots$$

where the (formal) power series $h(z) = 1 - \frac{1}{2}(z-1) + \frac{1}{3}(z-1)^2 - \ldots$ in fact converges at $z = 1$ with $h(1) = 1 = f(1)$. Therefore, $f(z) = h(z)$ on $B(1,1)$ and hence f is a power series on $B(1,1)$. By differentiability of power series, f is analytic at 1.

8. Prove that if f is analytic at z_0 and $f(z_0) = f'(z_0) = \cdots = f^{(m)}(z_0) = 0$, then the function g defined by means of the equations

$$g(z) = \begin{cases} \dfrac{f(z)}{(z-z_0)^{m+1}} & \text{when } z \neq z_0, \\ \dfrac{f^{(m+1)}(z_0)}{(m+1)!} & \text{when } z = z_0 \end{cases}$$

is analytic at z_0.

Solution. Since f is analytic at z_0, by definition, f is analytic on $B(z_0, r)$ for some $r > 0$. By Taylor's theorem, the Taylor Series of f centered at z_0 converges to itself on $B(z_0, r)$. Hence, for all $w \in B(z_0, r)$, we have

$$f(w) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)(w-z_0)^n}{n!} = \sum_{n=m+1}^{\infty} \frac{f^{(n)}(z_0)(w-z_0)^n}{n!}$$

Therefore in the deleted neighborhood $B(z_0, r)\backslash\{z_0\}$ of z_0, we have

$$g(z) = \frac{f(z)}{(z-z_0)^{m+1}} = \frac{\sum_{n=m+1}^{\infty} \frac{f^{(n)}(z_0)(z-z_0)^n}{n!}}{(z-z_0)^{m+1}} = \sum_{n=m+1}^{\infty} \frac{f^{(n)}(z-z_0)^{n-(m+1)}}{n!} = \sum_{n=0}^{\infty} \frac{f^{(n+m+1)}(z_0)(z-z_0)^n}{(n+m+1)!}$$

where the formal power series $h(z) := \sum_{n=0}^{\infty} \frac{f^{(n+m+1)}(z_0)(z-z_0)^n}{(n+m+1)!}$ converges at $z = z_0$ with $h(z_0) = \frac{f^{(m+1)}(z_0)}{(m+1)!} = g(z_0)$. Therefore, $g(z) = h(z)$ on $B(z_0, r)$ and hence g is a power series on $B(z_0, r)$. By differentiability of power series, f is analytic at z_0.

10. Consider two series

$$S_1(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{and} \quad S_2(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

which converge in some annular domain centered at z_0. Let C denote any contour lying in that annulus, and let $g(z)$ be a function which is continuous on C. Modify the proof of Theorem 1, Sec. 71, which tells us that

$$\int_C g(z) S_1(z)\, dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z - z_0)^n\, dz\,,$$

to prove that

$$\int_C g(z) S_2(z)\, dz = \sum_{n=1}^{\infty} b_n \int_C \frac{g(z)}{(z - z_0)^n}\, dz\,.$$

Conclude from these results that if

$$S(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

then

$$\int_C g(z) S(z)\, dz = \sum_{n=-\infty}^{\infty} c_n \int_C g(z)(z - z_0)^n\, dz\,.$$

Solution (Hints only).
First you should show that if a the formal series converges $f(z) := \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$ converges at some point $w \in \mathbb{C}$ where $w \neq z_0$, then f converges for all z with $|z - z_0| > |z_0 - w|$. In fact the latter converges absolutely.

Next consider the Annulus in question being $A(z_0, r, R) := \{z \in \mathbb{C} : r < |z - z_0| < R\}$ where $0 < r < R$. One should note that the contour C could be enclosed by a slightly smaller Annulus $A(z_0, r', R')$ where $r < r' < R' < R$ and the sum $\sum_{n=0}^{\infty} b_n \frac{g(z)}{(z-z_0)^n}$ converges uniformly on C. Explicitly we can do the following: since $A(z_0, r', R') \subsetneq A(z_0, r, R)$, we can take $z' \in A(z_0, r, R)$ with $\rho := |z' - z_0| \in (r, r')$. Then for all $z \in C$ and $n \in \mathbb{N}$, we have

$$\left| b_n \frac{g(z)}{(z - z_0)^n} \right| \leq |g(z)| \frac{\rho^n}{(z - z_0)^n} \frac{b_n}{\rho^n} \leq |g(z)| \left| \frac{b_n}{\rho^n} \right| \frac{\rho^n}{r'^n} \leq MN \frac{\rho^n}{r'^n}$$

where $M, N \in \mathbb{R}$ are independent of $z \in C$ such that $|g(z)| \leq M$ and $\left| \frac{b_n}{\rho^n} \right| \leq N$. The existence of the former is due to the extreme value theorem as g is continuous on C while the existence of the latter follows as $\left| \frac{b_n}{\rho^n} \right|$ is a bounded sequence as the series $\sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$ converges absolutely at $z = z'$ where $|z_0 - z'| = \rho$. The uniform convergence then follows from the convergence of the geometric series with ratio ρ/r' as $\frac{\rho}{r'} < 1$ and the fact that this convergence is independent of z.

After proving whose uniform convergence on the contour C, the integral-sum exchange formula follows similar as in the proof of the Taylor series.