MATH 2230A - HW 6 - Solutions

Full solutions at P.170-171 Q1, 3; P.177 Q1

Commonly missed steps in purple

For your reference, we recall the main theorems central to this HW. Note that the important Cauchy Integral Formula is derived from the Cauchy-Goursat Theorem on multiply-connected domain.

Theorem 0.1 (Cauchy Integral Formula). *Let $f : \Omega \to \mathbb{C}$ be holomorphic on a closed simply connected domain Ω. Let $z \in \Omega^o$, the interior of Ω, then we have*

$$2\pi i f(z) = \int_{\partial\Omega} \frac{f(w)}{w - z} dw$$

Theorem 0.2 (Generalized Cauchy Integral Formula). *Let $f : \Omega \to \mathbb{C}$ be holomorphic on a closed simply connected domain Ω. Then f is infinitely differentiable on Ω. Furthermore we have for all $n \in \mathbb{N}$ and $z \in \Omega^o$, the interior of Ω that*

$$\frac{1}{n!} f^{(n)}(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(w)}{(w - z)^{n+1}} dw$$

Corollary 0.3 (Liouville' Theorem). *Every bounded entire function is a constant function.*

Corollary 0.4 (Maximum Modulus Principle). *Let $f : \Omega \to \mathbb{C}$ be holomorphic on a closed connected set Ω. Suppose f attains maximum on an interior point of Ω, then f is a constant function.*

Remark. Every time, we would be giving full solutions to *selected* problems only. Other problems are provided with partial solutions. Please feel free to contact us if you need help on the solutions.

P.170 - 171

1. Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of these integrals:

(a) $\displaystyle\int_C \frac{e^{-z}\,dz}{z - (\pi i/2)}$; (b) $\displaystyle\int_C \frac{\cos z}{z(z^2+8)}\,dz$; (c) $\displaystyle\int_C \frac{z\,dz}{2z+1}$;

(d) $\displaystyle\int_C \frac{\cosh z}{z^4}\,dz$; (e) $\displaystyle\int_C \frac{\tan(z/2)}{(z - x_0)^2}\,dz \quad (-2 < x_0 < 2)$.

Ans. (a) 2π; (b) $\pi i/4$; (c) $-\pi i/2$; (d) 0; (e) $i\pi \sec^2(x_0/2)$.

Solution. We mainly use the Cauchy Integral Formula. To use it, we have to check that the function in target is holomorphic everywhere on the (simply connected) closed region enclosed by the boundary.

Denote Ω the closed region enclosed by C.

1. Let $f(z) = e^{-z}$. Define $z_0 := \pi i/2$. Then f is holomorphic on Ω and $z_0 \in \Omega^o$. By the Cauchy Integral Formula, we have

$$\int_C \frac{e^{-z}}{z - \pi i/2} dz = \int_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0) = 2\pi i e^{-\pi i/2} = 2\pi$$

2. Let $f(z) = \frac{\cos z}{z^2+8}$ and $z_0 := 0$. Then f is holomorphic on Ω (since its denominator is zerp if and only if $z = \pm 2\sqrt{2}i \notin \Omega$). Furthermore $0 = z_0 \in \Omega^o$. By the Cauchy Integral Formula, we have

$$\int_C \frac{\cos z}{z(z^2+8)} dz = \int_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0) = \frac{2\pi i}{8} = \frac{\pi i}{4}$$

3. Let $f(z) = z/2$ and $z_0 := -1/2$. Then $f(z)$ is clearly holomorphic on Ω and $z_0 \in \Omega^0$. By Cauchy Integral Formula, we have

$$\int_C \frac{z}{2z+1}dz = \int_C \frac{z/2}{z+1/2}dz = \int_C \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0) = -\frac{2\pi i}{4} = \frac{-\pi i}{2}$$

2. Find the value of the integral of $g(z)$ around the circle $|z-i| = 2$ in the positive sense when

(a) $g(z) = \dfrac{1}{z^2+4}$; (b) $g(z) = \dfrac{1}{(z^2+4)^2}$.

Ans. (a) $\pi/2$; (b) $\pi/16$.

Solution. For 2a, use the Cauchy Integral formula on $f(z) = \frac{1}{z+2i}$ at the point $z_0 := 2i$. Please do not forget to justify the choice of f and why the Cauchy Integral Formula can be used.
For 2b, use the Generalized Integral Formula on $f(z) = \frac{1}{(z+2i)^2}$ at the point $z_0 := 2i$.

3. Let C be the circle $|z| = 3$, described in the positive sense. Show that if

$$g(z) = \int_C \frac{2s^2 - s - 2}{s - z}\, ds \qquad (|z| \neq 3),$$

then $g(2) = 8\pi i$. What is the value of $g(z)$ when $|z| > 3$?

Solution. Denote Ω the closed region bounded by C, that is, the closed unit ball of radius 3. Let $f(s) = 2s^2 - s - 2$ and $z_0 := 2$. Then f is holomorphic on Ω and $z_0 \in \Omega^o$. By the Cauchy Integral Formula, we have

$$g(2) = \int_C \frac{2s^2 - s - 2}{s-2}ds = \int_C \frac{f(s)}{s-2}ds = 2\pi i f(2) = 2\pi i \cdot 4 = 8\pi i$$

Next, let $z \in \mathbb{C}$ be such that $|z| > 3$. Note that $z \notin \Omega^o$. We cannot use the Cauchy Integral Formula. Nonetheless, we can still use the Cauchy-Goursat Theorem: define $h_z(s) := \frac{2s^2-s-2}{s-z}$. Then h_z is holomorphic on Ω, which is simply connected, as $|z| > 3$. By the Cauchy-Goursat Theorem, we have

$$g(z) = \int_C \frac{2s^2 - s - 2}{s-z}ds = \int_C h_z(s)ds = 0$$

7. Let C be the unit circle $z = e^{i\theta}\ (-\pi \le \theta \le \pi)$. First show that for any real constant a,

$$\int_C \frac{e^{az}}{z}\, dz = 2\pi i.$$

Then write this integral in terms of θ to derive the integration formula

$$\int_0^\pi e^{a\cos\theta}\cos(a\sin\theta)\, d\theta = \pi.$$

Solution. For the first part, apply the cauchy integral formula on $f(z) = e^{az}$ at the point $z_0 = 0$. You should verify why the Cauchy Integral Formula could be used.
For the second part, simply the contour integral in the first part. You may have to consider odd functions/even functions.

P.177

1. Suppose that $f(z)$ is entire and that the harmonic function $u(x, y) = \text{Re}[f(z)]$ has an upper bound u_0; that is, $u(x, y) \leq u_0$ for all points (x, y) in the xy plane. Show that $u(x, y)$ must be constant throughout the plane.

Suggestion: Apply Liouville's theorem (Sec. 58) to the function $g(z) = \exp[f(z)]$.

Solution. The Liouville's Theorem states that a bounded entire function is a constant function. Let $g(z) := e^{f(z)}$. We proceed to show that g is a constant function. First since f is entire and the exponential function is entire, g is also entire by the chain rule. Then we proceed to show that g is bounded: let $z \in C$; then we have

$$|g(z)| = \left|e^{f(z)}\right| = \left|e^{u(z)+iv(z)}\right| = \left|e^{u(z)}\right| = e^{u(z)} \leq e^{u_0}$$

where u, v are real and imaginary parts of f respectively and the last two comparisons follow from the fact the the exponential function is non-negative and increasing on $\mathbb{R}$.
Hence, g is bounded by the constant e^{u_0} and is therefore bounded. By the Liouville's theorem, there exists $C \in \mathbb{C}$ such that $g(z) = C$ for all $z \in \mathbb{C}$. By considering modulus, we then have $e^{u(z)} = |e^{u(z)}| = |e^{f(z)}| = |g(z)| = |C|$ for all $z \in \mathbb{C}$. Note that $C \neq 0$ as the exponential function from complex domain always takes nonzero value (why?). Hence, $|C| \neq 0$. From the above, we then have $u(z) = \ln |C|$ for all $z \in C$, that is u is a constant on the plane.

4. Let R region $0 \leq x \leq \pi, 0 \leq y \leq 1$ (Fig. 75). Show that the modulus of the entire function $f(z) = \sin z$ has a maximum value in R at the boundary point $z = (\pi/2) + i$.

Suggestion: Write $|f(z)|^2 = \sin^2 x + \sinh^2 y$ (see Sec. 37) and locate points in R at which $\sin^2 x$ and $\sinh^2 y$ are the largest.

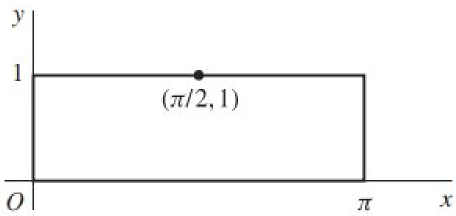

FIGURE 75

Solution. This question is simply a *verification* of the Maximum Modulus Principle. You probably do not have to apply any major theorems in this course. (Just following the suggestion will do).

6. Let f be the function $f(z) = e^z$ and R the rectangular region $0 \leq x \leq 1, 0 \leq y \leq \pi$. Illustrate results in Sec. 59 and Exercise 5 by finding points in R where the component function $u(x, y) = \text{Re}[f(z)]$ reaches its maximum and minimum values.

Ans. $z = 1, z = 1 + \pi i$.

Solution. This question is simply a *verification* of the Maximum Modulus Principle. You probably do not have to apply any major theorems in this course.