MATH 2230A - HW 5 - Solutions

Full solutions at Q1,2,4

For your reference, we recall the main theorems central to this HW.

Theorem 0.1 (Cauchy-Goursat Theorem). *Let Ω be the closure of some simply connected open set. Let $f : \Omega \to \mathbb{C}$ be a function such that it is holomorphic on Ω. Then we have*

$$\int_{\partial\Omega} f(z)dz = 0$$

where $\partial\Omega$ denotes the boundary of Ω

Theorem 0.2 (Cauchy-Goursat Theorem, multiply connected domain). *Let $f : \Omega \to \mathbb{C}$ be holomorphic on the closure of some open, multiply connected set, that is $\Omega = \overline{\Omega}_0 - \bigsqcup_{i=1}^{k} \Omega_i$ where $\Omega_0, \Omega_1, \ldots, \Omega_k$ are all open simply-connected sets and $\{\Omega_i\}_{1\le i\le k}$ are mutually disjoint subsets of Ω_0. Then we have*

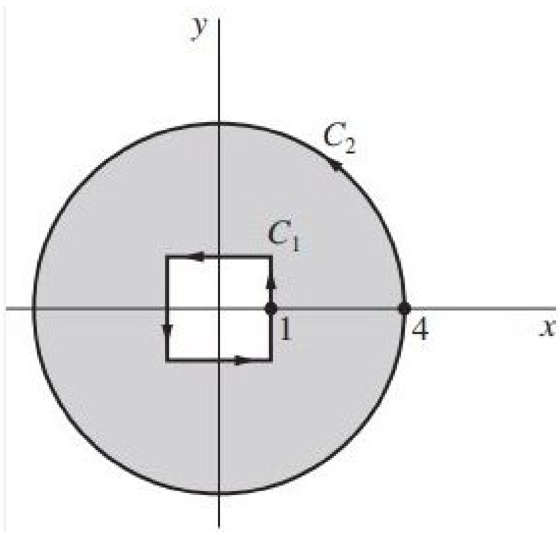
$$\int_{\partial\Omega_0} f(z)dz = \sum_{i=1}^{n} \int_{\partial\Omega_i} f(z)dz$$

where the boundaries are oriented couterclockwise.

Proof. This follows from the ordinary Cauchy-Goursat Theorem □

Remark. Every time, we would be giving full solutions to *selected* problems only. Other problems are provided with partial solutions. Please feel free to contact us if you need help on the solutions.

1 (P.159 Q2). Let C_1 denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 1, y = \pm 1$. Let C_2 be the positively oriented circle $|z| = 4$.



With the aid of Cauchy-Goursat Theorem on multiply connected domain, explain why $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$ when

a) $f(z) = \dfrac{1}{3z^2 + 1}$ b) $f(z) = \dfrac{z+2}{\sin(z/2)}$ c) $f(z) = \dfrac{z}{1 - e^z}$

Solution. We denote Ω the closed region between the two contours C_1, C_2. To use the Cauchy-Goursat Theorem, we have to ensure that the functions in questions are complex differentiable everywhere on Ω (including the boundary).

a). We consider the equation $3z^2+1=0$. Note that we have for all $z \in \mathbb{C}$

$$3z^2+1=0 \iff (\sqrt{3}z+i)(\sqrt{3}z-i)=0 \iff z=\pm\frac{i}{\sqrt{3}}$$

By the quotient rule, $f(z)$ is complex differentiable at z if the denominator $3z^2+1 \neq 0$ while f is not defined at z and so not differentiable at z if the denominator $3z^2+1=0$. Hence the set of complex differentiable points is $D_f := \mathbb{C} - \{\pm\frac{i}{\sqrt{3}}\}$. It is easy to see that $\Omega \subset D_f$, so f is complex differentiable on Ω and the Cauchy-Goursat Theorem applies.

b). First we consider the equation $\sin(z/2)=0$. Note that we have for all $z \in \mathbb{C}$

$$\sin(z/2)=0 \iff \frac{e^{iz/2}-e^{-iz/2}}{2i}=0 \iff e^{iz}-1=0 \iff iz \in \log 1 = \{2n\pi i | n \in \mathbb{Z}\}$$
$$\iff z = 2n\pi \text{ for some } n \in \mathbb{Z}$$

By the quotient rule, $f(z)$ is complex differentiable at z if the denominator $\sin(z/2) \neq 0$ while f is not defined at z and so not differentiable at z if the denominator $\sin(z/2)=0$. Hence the set of complex differentiable points is $D_f := \mathbb{C} - \{2n\pi | n \in \mathbb{Z}\}$. It is easy to see that $\Omega \subset D_f$, so f is complex differentiable on Ω and the Cauchy-Goursat Theorem applies.

c). First we consider the equation $1-e^z=0$. Note that we have for all $z \in \mathbb{C}$

$$1-e^z=0 \iff e^z=1 \iff z \in log 1 = \{2n\pi i | n \in \mathbb{Z}\}$$
$$\iff z = 2n\pi i \text{ for some } n \in \mathbb{Z}$$

By the quotient rule, $f(z)$ is complex differentiable at z if the denominator $1-e^z \neq 0$ while f is not defined at z and so not differentiable at z if the denominator $1-e^z=0$. Hence the set of complex differentiable points is $D_f := \mathbb{C} - \{2n\pi i | n \in \mathbb{Z}\}$. It is easy to see that $\Omega \subset D_f$, so f is complex differentiable on Ω and the Cauchy-Goursat Theorem applies.

2 (P.159 Q3)**.** Suppose C_0 is a positively oriented circle $|z-z_0|=R$ for some $R>0$ and $z_0 \in \mathbb{C}$, then it is true that

$$\int_{C_0} (z-z_0)^{n-1} dz = \begin{cases} 0 & 0 \neq n \in \mathbb{Z} \\ 2\pi i & n = 0 \end{cases}$$

Using the Cauchy-Goursat Theorem on multiply connected domain, show that if C is the boundary of the rectangle $0 \leq x \leq 3, 0 \leq y \leq 2$, oriented positively, then

$$\int_C (z-2-i)^{n-1} dz = \begin{cases} 0 & 0 \neq n \in \mathbb{Z} \\ 2\pi i & n = 0 \end{cases}$$

Solution. Consider a small circle centered at $(2+i)$ such that it lies in the interior of C. In particular, we take $C_0 = \{z \in \mathbb{C} : |z-(2+i)| < 0.1\}$, oriented positively. Then C_0 clearly lies in the interior of C. Let Ω be the closed region between the contour C_0 and C. To use the Cauchy-Goursat Theorem, we have to ensure that the functions $z \mapsto (z-2-i)^{n+1}$ for $n \in \mathbb{Z}$ is complex differentiable on Ω. This is true because by the quotient rule, the function is complex differentiable except maybe on the point $\{2+i\}$, which is not in Ω. Hence by the Cauchy-Goursat Theorem (on multiply connected domain), we have

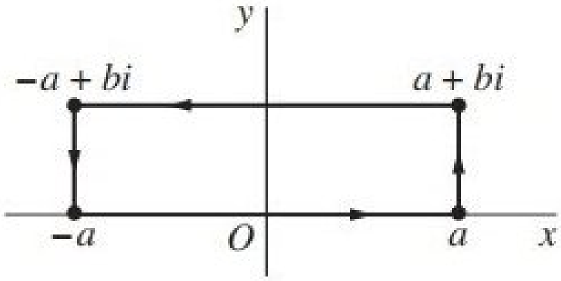
$$\int_C (z-2-i)^{n-1} dz = \int_{C_0} (z-2-i)^{n+1} = \begin{cases} 0 & 0 \neq n \in \mathbb{Z} \\ 2\pi i & n = 0 \end{cases}$$

where the second equality follows from the fact in the question.

3 (P.159 Q4)**.** In this question, we would be deriving the following integration formula for $b > 0$:

$$\int_0^\infty e^{-x^2} \cos 2bx dx = \frac{\sqrt{\pi}}{2} e^{-b^2}$$

(a). Consider the rectangular path below.



(i). Show that the sum of the integrals of e^{-z^2} along the lower and upper horizontal legs is

$$2\int_0^a e^{-x^2} dx - 2e^{b^2} \int_0^a e^{-x^2} \cos 2bx dx$$

(ii). Show that the sum of the integrals along the vertical legs on the right and left is

$$ie^{-a^2} \int_0^b e^{y^2} e^{-i2ay} dy - ie^{-a^2} \int_0^b e^{y^2} e^{i2ay} dy$$

(iii). With the aid of the Cauchy-Goursat theorem, show that

$$\int_0^a e^{-x^2} \cos 2bx dx = e^{-b^2} \int_0^a e^{-x^2} dx + e^{-(a^2+b^2)} \int_0^b e^{y^2} \sin 2ay dy$$

(b). By accepting the fact that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ and observing that

$$\left| \int_0^b e^{y^2} \sin 2ay dy \right| \leq \int_0^b e^{y^2} dy$$

show the desired integration formula at the start of the question by letting $a \to \infty$ in the equation at the end of part (a).

Solution. Just follow what the question tells you.

4 (P.161 Q6)**.** Let C denote the positively oriented boundary of the half disk $0 \le r \le 1, 0 \le \theta \le \pi$. Let f be a continuous function defined on the half disk by

$$f(z) = \begin{cases} 0 & z = 0 \\ \sqrt{r}e^{i\theta/2} & r > 0, -\frac{\pi}{2} < \theta \le \frac{3\pi}{2} \end{cases}$$

whenever z is in the half disk.

(a). By evaluating separately the integrals of f over the semi-circle and the two radii which made up C separately, show that $\int_C f(z)dz = 0$

(b). Why does the Cauchy-Goursat Theorem not apply here?

Solution.

(a). Let C_1, C_2, C_3 be the positively oriented contour of the semicircular arc, the left radius including the origin and the right radius including the origin respectively. Then $\int_C f(z)dz = \sum_{i=1}^{3} \int_{C_i} f(z)dz$ since C and $C_1 \cup C_2 \cup C_3$ are different by just a finite set of points and C_1, C_2, C_3 are almost disjoint from each other (by a finite set of points). We proceed to compute the integrals one by one:

On C_1. Parametrize C_1 by $\gamma_1 : [0, \pi] \to C_1$ with $\gamma_1(\theta) = e^{i\theta}$. Then we have

$$\begin{aligned} \int_{C_1} f(z)dz &= \int_0^\pi f \circ \gamma_1(\theta)\gamma_1'(\theta)d\theta = \int_0^\pi f(e^{i\theta})ie^{i\theta}d\theta \\ &= \int_0^\pi e^{i\theta/2}ie^{i\theta}d\theta \text{ since the branch is } -\frac{\pi}{2} < \theta \le \frac{3\pi}{2} \\ &= \int_0^\pi ie^{i3\theta/2}d\theta = \frac{2}{3i}ie^{i3\theta/2}\Big]_0^\pi = \frac{2}{3}(-i-1) \end{aligned}$$

On C_2. Parametrize C_2 by $\gamma_2 : [-1, 0] \to C_2$ with $\gamma_2(t) = t$. Then we have

$$\begin{aligned} \int_{C_2} f(z)dz &= \int_{-1}^0 f \circ \gamma_2(t)\gamma_2'(t)dt = \int_{-1}^0 f(t)dt \\ &= \lim_{\epsilon \to 0^-} \int_{-1}^{\epsilon} f(t)dt \\ &= \lim_{\epsilon \to 0^-} \int_{-1}^{\epsilon} (-t)^{1/2}e^{i\pi/2}dt = \lim_{\epsilon \to 0^-} -\frac{2}{3}(-t)^{3/2}i\Big]_{-1}^{\epsilon} = \frac{2}{3}i \end{aligned}$$

Note that we use improper integral on the 2nd row in case any problems appear at t = 0. Luckily, $f \circ \gamma_2(0) = f(0) = 0$ and $(-0)^{1/2}e^{i\pi/2} = 0 = f \circ \gamma_2(0)$. So for this case, improper integral needs not be used actually. Nonetheless, I expect some justifications similar to the above.

On C_3. Parametrize C_3 by $\gamma_3 : [0, 1] \to C_3$ with $\gamma_3(t) = t$. Then we have

$$\begin{aligned} \int_{C_3} f(z)dz &= \int_0^1 f \circ \gamma_3(t)\gamma_3'(t)dt = \int_0^1 f(t)dt \\ &= \lim_{\epsilon \to 0^+} \int_{\epsilon}^1 f(t)dt \\ &= \lim_{\epsilon \to 0^+} \int_{\epsilon}^1 t^{1/2}dt = \lim_{\epsilon \to 0^+} \frac{2}{3}t^{3/2}\Big]_{\epsilon}^1 = \frac{2}{3} \end{aligned}$$

Note that we use improper integral on the 2nd row in case any problems appear at t = 0. Luckily, $f \circ \gamma_3(0) = f(0) = 0$ and $(0)^{1/2} = 0 = f \circ \gamma_3(0)$. So for this case, improper integral needs not be used actually. Nonetheless, I expect some justifications similar to the above.

Therefore we have $\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz = 0$

(b). To use the Cauchy-Goursat Theroem, we have to show that the integrand is complex differentiable on the closed region enclosed by the contour of integration, which has to be the boundary of an open, simply-connected set. Now the contour C in question is the boundary of the simply connected upper half disk. We need to show that the integrand is not complex differentiable at some points on the closed region enclosed by C. We proceed to show that the integrand $f(z)$ is not complex differentiable at 0. Denote the closed region enclosed by C to be Ω. Note that for all $z \in \Omega - \{0\}$, we have

$$\frac{f(z) - f(0)}{z - 0} = \frac{f(z)}{z} = \frac{\sqrt{r}e^{i\theta/2}}{re^{i\theta}} = \frac{1}{\sqrt{r}}e^{-i\theta/2}$$

So, if we take a sequence $z_n \to 0$ with $\theta = 0$ and decreasing r with $z_n \in \Omega$, for example, $z_n = 1/n$ on the real line. Then we have

$$\frac{f(z_n) - f(0)}{z_n - 0} = \frac{f(z_n)}{z_n} = \sqrt{n}$$

which shows that z_n is unbounded and thus does not converge. By sequential criteria $\lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$ does not exist, that is, f is not complex differentiable at 0.
Since the condition of the Cauchy-Goursat Theorem is not fulfilled, we cannot apply it here (at least directly).

5 (P.161 Q7)**.** Show that for any positively oriented simple closed contour C, the area of the region enclosed by C can be written as

$$\frac{1}{2i}\int_C \overline{z}dz$$

Hint: The Green Theorem you learnt in multivariable calculus may be useful. You may consult Sec.50 of the textbook

Solution. Please refer to similar examples in Sec.50 of the textbook for the method.