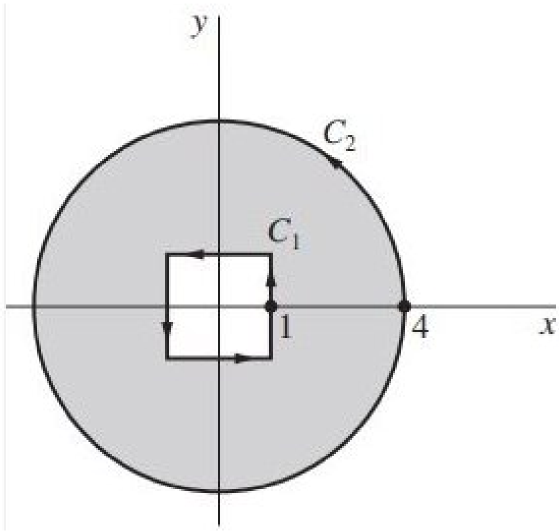
MATH 2230A - HW 5 - Comments and Common Mistakes:

Common Misktakes:

Question 1:

Let C_1 denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 1, y = \pm 1$. Let C_2 be the positively oriented circle $|z| = 4$.



With the aid of Cauchy-Goursat Theorem on multiply connected domain, explain why $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$ when

a) $f(z) = \dfrac{1}{3z^2+1}$ b) $f(z) = \dfrac{z+2}{\sin(z/2)}$ c) $f(z) = \dfrac{z}{1-e^z}$

1. The main problem is on Q1c.

1c) $f(z) = \frac{z}{1-e^z}$ is analytic everywhere except when $1-e^z=0$

$\Rightarrow$ $z=0$ inside C_1

f is analytic everywhere in $\mathbb{C}$ except when $1-e^z=0$.

$1-e^z=0$

$\Rightarrow e^z=1$

$z=0$.

Many of you were trapped when solving equations involving exponential/ trigonometric functions: there are always infinitely many solutions as they are not injective with domain being the complex plane.

Question 2:

Suppose C_0 is a positively oriented circle $|z - z_0| = R$ for some $R > 0$ and $z_0 \in \mathbb{C}$, then it is true that

$$\int_{C_0} (z - z_0)^{n-1} dz = \begin{cases} 0 & 0 \neq n \in \mathbb{Z} \\ 2\pi i & n = 0 \end{cases}$$

Using the Cauchy-Goursat Theorem on multiply connected domain, show that if C is the boundary of the rectangle $0 \le x \le 3, 0 \le y \le 2$, oriented positively, then

$$\int_{C} (z - 2 - i)^{n-1} dz = \begin{cases} 0 & 0 \neq n \in \mathbb{Z} \\ 2\pi i & n = 0 \end{cases}$$

1. The performance of this question is good. You can either enclose the rectangle in questions by a very large circle in its *interior* or enclose a very small circle within the *interior* of the rectangle that contains the singularity $2 + i$.

Question 4:

Let C denote the positively oriented boundary of the half disk $0 \le r \le 1, 0 \le \theta \le \pi$. Let f be a continuous function defined on the half disk by

$$f(z) = \begin{cases} 0 & z = 0 \\ \sqrt{r} e^{i\theta/2} & r > 0, -\frac{\pi}{2} < \theta \le \frac{3\pi}{2} \end{cases}$$

whenever z is in the half disk.

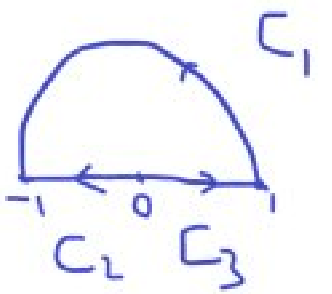
(a). By evaluating separately the integrals of f over the semi-circle and the two radii which made up C separately, show that $\int_C f(z)dz = 0$

(b). Why does the Cauchy-Goursat Theorem not apply here?

1. Some students have misread the question and used a wrong parametrization: please determine why the parametrizations below are wrong.

4a(1) Parametrice by $\gamma_1 : [0,1] \to \mathbb{C}$ given by $\gamma_1(t) = t$.
Parametrize by $\gamma_2 : [0,\pi] \to \mathbb{C}$, given by $\gamma_2(t) = \frac{1}{2} + \frac{1}{2}e^{it}$

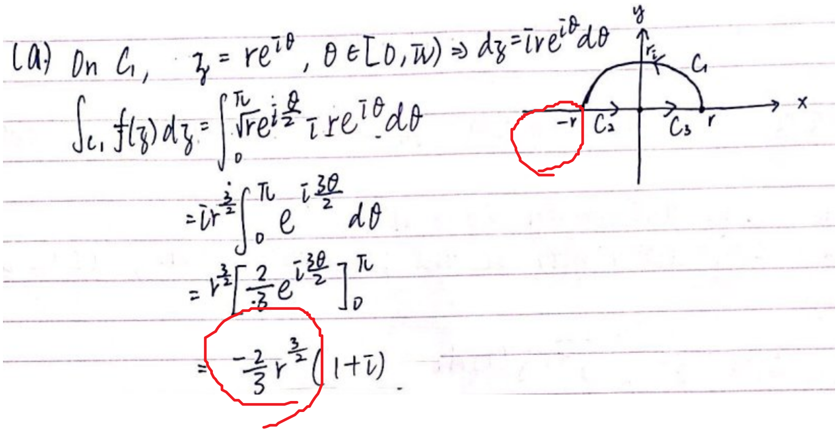
Student 1



Student 2

Answer: 1. The parametrization in question is the unit half disk centered at the origin with radius 1. The parametrization here is one with radius $1/2$ and cetered at $1/2$. You can check that the argument of the points in the contour does not match: the points on the latter lie on the first quadrant so the arguments of the points lie in $[0, \pi/2]$, not $[0, \pi]$ as in the question. 2. Wrong orientation..

2. Some students misunderstood the r in the definition of the function.

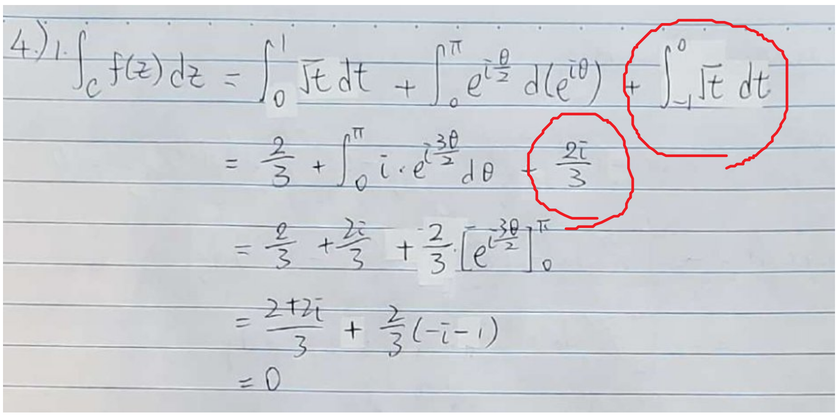


(a) On C_1, $z = re^{i\theta}$, $\theta \in [0, \pi) \Rightarrow dz = ire^{i\theta} d\theta$

$$\int_{C_1} f(z)\,dz = \int_0^\pi \sqrt{r}e^{i\frac{\theta}{2}}\, ire^{i\theta} d\theta = ir^{\frac{3}{2}} \int_0^\pi e^{i\frac{3\theta}{2}} d\theta = r^{\frac{3}{2}}\left[\frac{2}{3}e^{i\frac{3\theta}{2}}\right]_0^\pi = -\frac{2}{3}r^{\frac{3}{2}}(1+i)$$

This r stands for the modulus of the input variable, but not an arbitrary constant. In this course, we would be abusing notation to use x, y, r, θ,etc when defining functions to denote the real part, imaginary part, modulus, argument, etc of the input variable unless otherwise specified. Indeed there is no constant r defined outside the function definition; that should give you hints on what the r means in the question.

3. Some of you have misunderstood the process of evaluating contour integrals:



4).1. $\int_C f(z)\,dz = \int_0^1 \sqrt{t}\,dt + \int_0^\pi e^{i\frac{\theta}{2}}\, d(e^{i\theta}) + \int_{-1}^0 \sqrt{t}\,dt$

$= \frac{2}{3} + \int_0^\pi i \cdot e^{i\frac{3\theta}{2}} d\theta + \frac{2i}{3}$

$= \frac{2}{3} + \frac{2i}{3} + \frac{2}{3}\left[e^{i\frac{3\theta}{2}}\right]_0^\pi$

$= \frac{2+2i}{3} + \frac{2}{3}(-i-1)$

$= 0$

the integral you get after you parametrizing your contour is an integral on a *real* interval, which is evaluated by considering the real and imaginary part of the integrand. Therefore, to evaluate such integral, you should typically first write the integrand into real-valued functions by considering the real/imaginary parts. Before which, you cannot apply the anti-derivative technique (which follows from the Fundamental Theorem on Calculus on $\mathbb{R}$). There is an exception: you can use the anti-derivative technique if you consider the real interval domain to be embedded as a straight-lined contour in $\mathbb{C}$ and apply the path-independence on the integral. Nonetheless, if you go this way, I expect you to state clear the conditions on which you can apply the above result (which was central to HW4). In fact, I think the student here did not really put a parametrization on the contour, which is the left radius. Please read the solution on how to do contour integrals correctly.