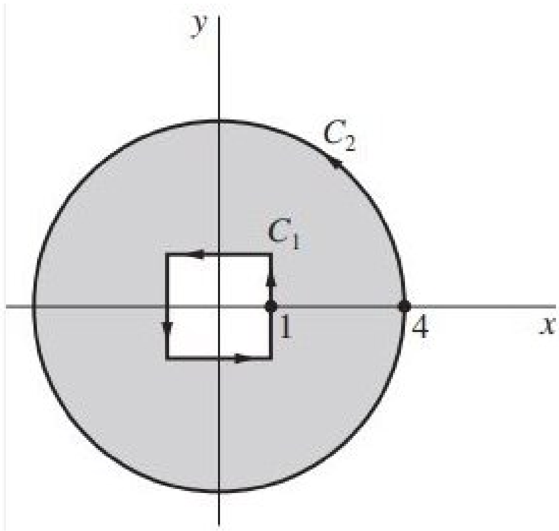
MATH 2230A - HW 5
Due Date: 23 Oct 2020 (Fri), 23:59
(Please change the file name of your HW pdf to **HW#_1155xxxxxx_ChaiTaiMan***)*

Problems: P.159-161: Q2,3,4,6,7
(5 Questions in total)

Textbook: Brown JW, Churchill RV(2014). Complex Variables and Applications, nineth edition, McGraw-Hill Education

We type here all the required problems *for your convenience only.* The presentation of the problems here may be different from the original one but the respective solution should be unaffected.

1 (P.159 Q2)**.** Let C_1 denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 1, y = \pm 1$. Let C_2 be the positively oriented circle $|z| = 4$.



With the aid of Cauchy-Goursat Theorem on multiply connected domain, explain why $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$ when

a) $f(z) = \dfrac{1}{3z^2 + 1}$ b) $f(z) = \dfrac{z + 2}{\sin(z/2)}$ c) $f(z) = \dfrac{z}{1 - e^z}$

2 (P.159 Q3)**.** Suppose C_0 is a positively oriented circle $|z - z_0| = R$ for some $R > 0$ and $z_0 \in \mathbb{C}$, then it is true that

$$\int_{C_0} (z - z_0)^{n-1} dz = \begin{cases} 0 & 0 \neq n \in \mathbb{Z} \\ 2\pi i & n = 0 \end{cases}$$

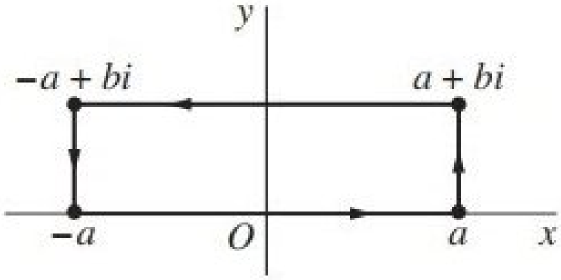
Using the Cauchy-Goursat Theorem on multiply connected domain, show that if C is the boundary of the rectangle $0 \leq x \leq 3, 0 \leq y \leq 2$, oriented positively, then

$$\int_{C} (z - 2 - i)^{n-1} dz = \begin{cases} 0 & 0 \neq n \in \mathbb{Z} \\ 2\pi i & n = 0 \end{cases}$$

3 (P.159 Q4). In this question, we would be deriving the following integration formula for $b > 0$:

$$\int_0^\infty e^{-x^2} \cos 2bx dx = \frac{\sqrt{\pi}}{2} e^{-b^2}$$

(a). Consider the rectangular path below.



(i). Show that the sum of the integrals of e^{-z^2} along the lower and upper horizontal legs is

$$2\int_0^a e^{-x^2} dx - 2e^{b^2} \int_0^a e^{-x^2} \cos 2bx dx$$

(ii). Show that the sum of the integrals along the vertical legs on the right and left is

$$ie^{-a^2} \int_0^b e^{y^2} e^{-i2ay} dy - ie^{-a^2} \int_0^b e^{y^2} e^{i2ay} dy$$

(iii). With the aid of the Cauchy-Goursat theorem, show that

$$\int_0^a e^{-x^2} \cos 2bx dx = e^{-b^2} \int_0^a e^{-x^2} dx + e^{-(a^2+b^2)} \int_0^b e^{y^2} \sin 2ay dy$$

(b). By accepting the fact that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ and observing that

$$\left| \int_0^b e^{y^2} \sin 2ay dy \right| \leq \int_0^b e^{y^2} dy$$

show the desired integration formula at the start of the question by letting $a \to \infty$ in the equation at the end of part (a).

4 (P.161 Q6). Let C denote the positively oriented boundary of the half disk $0 \leq r \leq 1, 0 \leq \theta \leq \pi$. Let f be a continuous function defined on the half disk by

$$f(z) = \begin{cases} 0 & z = 0 \\ \sqrt{r} e^{i\theta/2} & r > 0, -\frac{\pi}{2} < \theta \leq \frac{3\pi}{2} \end{cases}$$

whenever z is in the half disk.

1. By evaluating separately the integrals of f over the semi-circle and the two radii which made up C separately, show that $\int_C f(z)dz = 0$
2. Why does the Cauchy-Goursat Theorem not apply here?

5 (P.161 Q7). Show that for any positively oriented simple closed contour C, the area of the region enclosed by C can be written as

$$\frac{1}{2i} \int_C \overline{z} dz$$

Hint: The Green Theorem you learnt in multivariable calculus may be useful. You may consult Sec.50 of the textbook