MATH 2230A - HW 4 - Solutions

Full solutions at Q5,7,11

Remark. Every time, we would be giving full solutions to *selected* problems only. Other problems are provided with partial solutions. Please feel free to contact us if you need help on the solutions.

1 (P.119 Q2)**.** Evaluate the following integrals.

a) $\int_0^1 (1+it)^2 dt$

b) $\int_1^2 (\frac{1}{t} - i)^2 dt$

c) $\int_0^{\pi/6} e^{i2t} dt$

d) $\int_0^\infty e^{-zt} dt$ where $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$

Solution. Note that the integrands are of real domains. Use the *definition* for integrals of complex-valued functions with real domain to compute the integrals.

2 (P.119 Q3)**.** Let $m, n \in \mathbb{Z}$. Show that

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \begin{cases} 0 & m \neq n \\ 2\pi & m = n \end{cases}$$

Solution. Note that the integrands are of real domains. Use the *definition* for integrals of complex-valued functions with real domain to compute the integrals.

3 (P.119 Q4)**.** Consider the following identity:

$$\int_0^\pi e^{(1+i)x} dx = \int_0^\pi e^x \cos x dx + i \int_0^\pi e^x \sin x dx$$

Evalute the two real-valued integrals on the right by considering the real and imaginary parts of the complex-valued integral on the left.

Solution. Just do what the question asks you.

4 (P.133 Q3,4,6,7,8)**.** For each of the following complex functions f and contours C, evaluate by parametrizing C the contour integral,

$$\int_C f(z) dz$$

1. $f(z) = \pi \exp(\pi \bar{z})$. C is the boundary of the square with vectices $0, 1, i, 1+i$, oriented counter clockwise.
2. $f(z) = \begin{cases} 1 & y < 0 \\ 4y & y > 0 \end{cases}$. C is the arc from $-1-i$ to $1+i$ along the curve $y = x^3$.
3. $f(z) = z^i$ where the principal branch is used. C is the semi-circle $z = e^{i\theta}$, $0 \leq \theta \leq \pi$
4. $f(z) = z^{-1-2i}$ using the principal branch. C is the contour $z = e^{i\theta}$ where $0 \leq \theta \leq \pi/2$
5. $f(z) = z^{a-1}$ using the principal branch with $0 \neq a \in \mathbb{R}$. C is the positively oriented circle of radius R about the origin.

Solution. Use the definition of contour integrals to compute. Note that for Q2, the integrand is discontinuous on the path. Improper integral has to be used.

Remark. An integral is still well-defined if the integrand is un-defined for a finite point on the contour. Nonetheless, some techniques may not be applicable.

5 (P.133 Q9)**.** Let C denote the positively oriented unit circle $|z| = 1$ about the origin.

(a). Let $f(z) = z^{-3/4}$ using the principal branch. Show that $\int_C f(z)dz = 4\sqrt{2}i$

(b). Let $g(z) = z^{-3/4}$ using the branch $0 < \arg z \leq 2\pi$. Show that $\int_C g(z)dz = -4 + 4i$

Solution.

1. Note that $f(z) = e^{-3/4ln|z|}e^{-3/4i\,\mathrm{Arg}(z)} = e^{-3/4iArg(z)}$ for $z \in C$ where $\mathrm{Arg}(z) \in (-\pi, \pi]$. Parametrize C with $[-\pi, \pi]$ by $\theta \mapsto e^{i\theta}$. Then we have

$$\begin{aligned}\int_C f(z)dz &= \int_{-\pi}^{\pi} e^{-3/4i\,\mathrm{Arg}(e^{i\theta})} ie^{i\theta} d\theta = \lim_{t\to-\pi^+} \int_t^{\pi} e^{-3/4iArg(e^{i\theta})} ie^{i\theta} d\theta \\ &= \lim_{t\to-\pi^+} \int_t^{\pi} e^{-3/4i\theta} ie^{i\theta} d\theta = \lim_{t\to-\pi^+} \int_t^{\pi} e^{i\theta/4} id\theta \\ &= \lim_{t\to-\pi^+} \frac{4}{i} ie^{i\theta/4}\Big]_t^{\pi} = 4(e^{i\pi/4} - e^{-i\pi/4}) = 4 \cdot 2i \cdot \sin(\pi/4) = 4\sqrt{2}i\end{aligned}$$

   Improper integral has to be used since the integrand is not continuous at an endpoint. Otherwise, the Fundamental Theorem of Calculus cannot be used.

2. Note that $f(z) = e^{-3/4ln|z|}e^{-3/4i\arg(z)} = e^{-3/4i\arg(z)}$ for $z \in C$ where $\arg(z) \in (-0, 2\pi]$. Parametrize C with $[0, 2\pi]$ by $\theta \mapsto e^{i\theta}$. Then we have

$$\begin{aligned}\int_C f(z)dz &= \int_0^{2\pi} e^{-3/4i\arg(e^{i\theta})} ie^{i\theta} d\theta = \lim_{t\to 0^+} \int_t^{2\pi} ie^{-3/4i\arg(e^{i\theta})} ie^{i\theta} d\theta \\ &= \lim_{t\to 0^+} \int_t^{2\pi} e^{-3/4i\theta} ie^{i\theta} d\theta = \lim_{t\to 0^+} \int_t^{2\pi} e^{i\theta/4} id\theta \\ &= \lim_{t\to 0^+} \frac{4}{i} ie^{i\theta/4}\Big]_t^{2\pi} = 4(e^{i\pi/2} - e^{i0}) = 4i - 4\end{aligned}$$

   Improper integral has to be used since the integrand is not continuous at an endpoint. This part is basically the same as the previous part.

Remark. Please note that in the integrand of each of the contour integral, the argument function is NOT evaluated until the contour has been adjusted to one on which the argument function $\arg(e^{i\theta})$ agrees at ALL points (including endpoints) with its evaluation θ.

6 (P.134 Q11)**.** Let C denote the semicircular arc from $-2i$ to $2i$ of the circle $|z| = 2$, oriented counter-clockwise. Let $f(z) = \overline{z}$. Evaluate the contour integral $\int_C f(z)dz$ using the following parametrizations of C

1. $z = 2e^{i\theta}$ where $\pi/2 \leq \theta \leq \pi/2$
2. $z = \sqrt{4 - y^2} + iy$ where $-2 \leq y \leq 2$

Solution. Use the definition of contour integral to compute.

7 (P.139 Q4). Let $R > 2$. Denote the *upper half* of the circle $|z| = R$, C_R, oriented counter-clockwise.

1. Show that
$$\left| \int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4} dz \right| \le \frac{\pi R(2R^2 + 1)}{(R^2 - 1)(R^2 - 4)}$$

2. Show that the value of the integral tends to 0 as $R \to \infty$.
   Hint: Divide the numerator and denominator on the right by R^4.

Solution.

1. Note that when $z \in C_R$, by triangle inequality, we have
$$\begin{aligned} \left|2z^1 - 1\right| &\le \left|2z^2\right| + |-1| = 2R^2 + 1 \\ \left|z^2 + 1\right| &\ge \left|\left|z^2\right| - |-1|\right| = R^2 - 1 \\ \left|z^2 + 1\right| &\ge \left|\left|z^2\right| - |-4|\right| = R^2 - 4 \end{aligned}$$

   where the condition $R > 2$ has been used for the last two inequality. Therefore, on C_R
$$\left| \frac{2z^2 - 1}{z^4 + 5z^2 + 4} \right| = \frac{\left|2z^2 - 1\right|}{|z^4 + 5z^2 + 4|} \le \frac{2R^2 + 1}{(R^2 - 1)(R^2 - 4)}$$
   By the modulus inequality for integral, we have
$$\begin{aligned} \left| \int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4} dz \right| &\le \int_{C_R} \left| \frac{2z^2 - 1}{z^4 + 5z^2 + 4} \right| |dz| \le \int_{C_R} \frac{(2R^2 + 1)}{(R^2 - 1)(R^2 - 4)} |dz| \\ &= \frac{(2R^2 + 1)}{(R^2 - 1)(R^2 - 4)} \int_{C_R} |dz| = \frac{\pi R(2R^2 + 1)}{(R^2 - 1)(R^2 - 4)} \end{aligned}$$

2. Note that we have
$$\frac{\pi R(2R^2 + 1)}{(R^2 - 1)(R^2 - 4)} = \frac{(2 + \frac{1}{R^2})\frac{1}{R}}{(1 - \frac{1}{R^2})(1 - \frac{4}{R^2})} \to \frac{(2 + 0) \cdot 0}{(1 - 0)(1 - 0)} = 0$$
   as $R \to \infty$.
   Hence by squeeze theorem, the modulus of the integral converges to 0 as $R \to \infty$ and so as the integral itself.

8 (P.139 Q5). Let $R > 1$. Let C_R be the circle $|z| = R$, oriented counter-clockwise.

1. Show that
$$\left| \int_{C_R} \frac{z}{z^2} dz \right| \le 2\pi \left(\frac{\pi + \ln R}{R} \right)$$

2. Show that the integral tends to zero as $R \to \infty$ by l'Hosptial's Rule.

Solution. It is similar to Q7. You can still apply the modulus inequality for integral here as long as the integrand is piecewise continuous on the path.

9 (P.139 Q6)**.** Let $\rho \in (0, 1)$. Let C_ρ denot the circle $|z| = \rho$, oriented counterclockwise. Let f be a function analytic on the disk $|z| \le 1$. Consider the power function $z^{-1/2}$ using the *any* particular branch.

1. Show that there exists a real constant $M > 0$ independent of ρ such that

$$\left| \int_{C_\rho} z^{-1/2} f(z) dz \right| \le 2\pi M \sqrt{\rho}$$

2. Hence, show that the value to the integral approaches 0 as $\rho \to 0$.

Hint: Use the Extreme Value Theorem on $\mathbb{C}$: a continuous function on a closed and bounded set $\Omega \subset \mathbb{C}$ attains maximum and hence is bounded there

Solution. It is similar to Q7. You can still apply the modulus inequality for integral here as long as the integrand is piecewise continuous on the path. The M in the question can be taken to be any bound of $f(z)$ on $\{z \in \mathbb{C} : |z| \le 1\} \supset C_\rho$, which exists due to the extreme value theorem as the closed unit disk $\{z \in \mathbb{C} : |z| \le 1\}$ is closed and bounded and analyticity (of f) implies continuity.

10 (P.147 Q2)**.** Evaluate each of the following integrals, where the path is any contours between the indicated endpoints, by finding an anti-derivative.

a) $\int_0^{1+i} z^2 dz$ b) $\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right) dz$ c) $\int_1^3 (z-2)^3 dz$

Solution. Antiderivatives of these integrands exist everywhere on $\mathbb{C}$. Taking any path you like (which is continuously differentiable) will do.

11 (P.147 Q5)**.** Show that

$$\int_{-1}^{1} z^i dz = \frac{1 + e^{-\pi}}{2}(1 - i)$$

where z^i uses the principal branch and the path of integration is any contour from -1 to 1 that lies above the real axis (except for its endpoints).

Solution. Let $f(z) = z^i$ using the principle branch ($-\pi < \arg z \le \pi$) and $g(z) = z^i$ using the branch $-\pi/2 < \arg z \le 3\pi/2$.
Let C be any contour from -1 to 1 that lies above the real axis. Note that for any z on the upper half plane, $f(z) = g(z)$ since we take $\arg z \in (0, \pi)$ when evaluating both functions. Therefore, $f = g$ on C except maybe for the endpoints, which is a *finite set of points* (so $f = g$ *"almost everywhere"* on C). Therefore, we have $\int_C f(z)dz = \int_C g(z)dz$.
Since $g(z)$ is continuous and has an anti-derivative everywhere except on its branch cut and C lies on the (interior) of the complement of the branch cut (hence lies on some *open connected* sets on which g has an antiderivative). Therefore, by the Fundamental Theorem of Contour Integral (Theorem 1.34 in the Lecture Note), we have

$$\begin{aligned} \int_C f(z)dz = \int_C g(z)dz &= \frac{1}{1+i} z^{1+i} \Big]_{-1}^{1} = \frac{1}{1+i} e^{(i+1)\ln|z|} e^{i(i+1)\arg z} \Big]_{-1}^{1} \\ &= \frac{1}{i+1}(e^0 - e^{i(i+1)\pi}) = \frac{1}{i+1}(1 - e^{-\pi} e^{i\pi}) = \frac{1-i}{2}(1 + e^{-\pi}) \end{aligned}$$

The result follows from the above computation.