MATH 2230A - HW 4
Due Date: 16 Oct 2020 (Fri), 23:59
(Please change the file name of your HW pdf to **HW#_1155xxxxxx_ChaiTaiMan***)*

Problems: P.119: Q2,3,4; P.133: Q3,4,6,7,8,9; P. 134: Q11; P. 139: Q4,5,6; P.147: Q2,5
(15 Questions in total)

Textbook: Brown JW, Churchill RV(2014). Complex Variables and Applications, nineth edition, McGraw-Hill Education

We type here all the required problems *for your convenience only.* The presentation of the problems here may be different from the original one but the respective solution should be unaffected.

Remark. The principal branch always implies $-\pi < \arg z \leq \pi$. Unless otherwise specified, the symbols x, y should denote the real and imaginary parts of a complex number z.

1 (P.119 Q2)**.** Evaluate the following integrals.

a) $\int_0^1 (1+it)^2 dt$

b) $\int_1^2 (\frac{1}{t} - i)^2 dt$

c) $\int_0^{\pi/6} e^{i2t} dt$

d) $\int_0^{\infty} e^{-zt} dt$ where $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$

2 (P.119 Q3)**.** Let $m, n \in \mathbb{Z}$. Show that

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \begin{cases} 0 & m \neq n \\ 2\pi & m = n \end{cases}$$

3 (P.119 Q4)**.** Consider the following identity:

$$\int_0^{\pi} e^{(1+i)x} dx = \int_0^{\pi} e^x \cos x dx + i \int_0^{\pi} e^x \sin x dx$$

Evalute the two real-valued integrals on the right by considering the real and imaginary parts of the complex-valued integral on the left.

4 (P.133 Q3,4,6,7,8)**.** For each of the following complex functions f and contours C, evaluate by parametrizing C the contour integral,

$$\int_C f(z) dz$$

1. $f(z) = \pi \exp(\pi \bar{z})$. C is the boundary of the square with vectices $0, 1, i, 1+i$, oriented counter clockwise.
2. $f(z) = \begin{cases} 1 & y < 0 \\ 4y & y > 0 \end{cases}$. C is the arc from $-1-i$ to $1+i$ along the curve $y = x^3$.
3. $f(z) = z^i$ where the principal branch is used. C is the semi-circle $z = e^{i\theta}$, $0 \leq \theta \leq \pi$
4. $f(z) = z^{-1-2i}$ using the principal branch. C is the contour $z = e^{i\theta}$ where $0 \leq \theta \leq \pi/2$
5. $f(z) = z^{a-1}$ using the principal branch with $0 \neq a \in \mathbb{R}$. C is the positively oriented circle of radius R about the origin.

Remark. An integral is still well-defined if the integrand is un-defined for a finite point on the contour. Nonetheless, some techniques may not be applicable.

5 (P.133 Q9). Let C denote the positively oriented unit circle $|z| = 1$ about the origin.

(a). Let $f(z) = z^{-3/4}$ using the principal branch. Show that $\int_C f(z)dz = 4\sqrt{2}i$

(b). Let $g(z) = z^{-3/4}$ using the branch $0 < \arg z < 2\pi$. Show that $\int_C g(z)dz = -4 + 4i$

6 (P.134 Q11). Let C denote the semicircular arc from $-2i$ to $2i$ of the circle $|z| = 2$, oriented counter-clockwise. Let $f(z) = \overline{z}$. Evaluate the contour integral $\int_C f(z)dz$ using the following parametrizations of C

1. $z = 2e^{i\theta}$ where $\pi/2 \leq \theta \leq \pi/2$
2. $z = \sqrt{4 - y^2} + iy$ where $-2 \leq y \leq 2$

7 (P.139 Q4). Let $R > 2$. Denote the *upper half* of the circle $|z| = R$, C_R, oriented counter-clockwise.

1. Show that
$$\left| \int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4} dz \right| \leq \frac{\pi R(2R^2 + 1)}{(R^2 - 1)(R^2 - 4)}$$
2. Show that the value of the integral tends to 0 as $R \to \infty$.
Hint: Divide the numerator and denominator on the right by R^4.

8 (P.139 Q5). Let $R > 1$. Let C_R be the circle $|z| = R$, oriented counter-clockwise.

1. Show that
$$\left| \int_{C_R} \frac{\operatorname{Log} z}{z^2} dz \right| \leq 2\pi \left(\frac{\pi + \ln R}{R} \right)$$
2. Show that the integral tends to zero as $R \to \infty$ by l'Hosptial's Rule.

9 (P.139 Q6). Let $\rho \in (0, 1)$. Let C_ρ denot the circle $|z| = \rho$, oriented counterclockwise. Let f be a function analytic on the disk $|z| \leq 1$. Consider the power function $z^{-1/2}$ using the *any* particular branch.

1. Show that there exists a real constant $M > 0$ independent of ρ such that
$$\left| \int_{C_\rho} z^{-1/2} f(z) dz \right| \leq 2\pi M \sqrt{\rho}$$
2. Hence, show that the value to the integral approaches 0 as $\rho \to 0$.

Hint: Use the Extreme Value Theorem on $\mathbb{C}$*: a continuous function on a closed and bounded set* $\Omega \subset \mathbb{C}$ *attains maximum and hence is bounded there*

10 (P.147 Q2). Evaluate each of the following integrals, where the path is any contours between the indicated endpoints, by finding an anti-derivative.

a) $\int_0^{1+i} z^2 dz$ b) $\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right) dz$ c) $\int_1^3 (z-2)^3 dz$

11 (P.147 Q5). Show that
$$\int_{-1}^{1} z^i dz = \frac{1 + e^{-\pi}}{2}(1 - i)$$
where z^i uses the principal branch and the path of integration is any contour from -1 to 1 that lies above the real axis (except for its endpoints).