MATH 2230A - HW 11 - Related Facts and Solutions

Below are key definitions and theorems relating to this homework. All curves are oriented counter clockwise.

Lemma 0.1 (Jordan's Inequality). *Denote $C_R := \{z \in \mathbb{C} : |z| = R, \mathrm{Im}(z) \geq 0\}$, that is the upper semi-circle of radius R. Let f be a function continuous on C_R. Let $a > 0$ be a real constant. Then we have*

$$\left|\int_{C_R} f(z)e^{iaz}dz\right| \leq \frac{\pi}{a}M_R$$

where $M_R := \sup_{z\in C_R} |f(z)|$.

Remark.

- The proofs in fact follow from triangle inequality of integrals
- In the proof, the concavity of the sine function on $[0, \pi/2]$ is used.
- This approximation is important because the right-hand side is independent of R.

Theorem 0.2 (Jordan's Lemma). *Denote $C_R := \{z \in \mathbb{C} : |z| = R, \mathrm{Im}(z) \geq 0\}$. Let f be a function. Suppose f is continuous on C_R when R is sufficiently large, that is there exists $R_0 > 0$ such that f is continuous on C_R when $R \geq R_0$. Suppose further $M_R := \sup_{z\in C_R} |f(z)| \to 0$ as $R \to \infty$. Then*

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{iaz}dz = 0.$$

where $a > 0$

Remark. This follows directly from the Jordan's inequality.

Theorem 0.3 (Indented path approximation, Sec.89 in Textbook). *Let $z_0 \in \mathbb{C}$. Let f be holomorphic on a deleted neighbhorhood of x_0. Suppose x_0 is a simple pole of f. Denote $C_\rho := \{z \in \mathbb{C} : |z - z_0| = \rho, \mathrm{Im}(z - z_0) \geq 0\}$, that is, the upper semicircular arc of radius ρ centered at x_0. Then we have*

$$\lim_{\rho\to0} \int_{C_\rho} f(z)dz = i\pi \,\mathrm{Res}(f, z_0)$$

Remark. We mostly consider the case $z_0 = 0$ or on the real-axis. The limit follows by considering the Laurent Series of f at z_0. The holomorphic part converges to 0 because it is bounded on closed disks by the extreme value theorem.

The above facts are useful for the first 13 questions. We shall see next the related definitions and theorems for the last 4 questions on the next page.

Definition 0.4 (Meromorphic functions)**.** Let Ω be a (open connected) domain. Then a function $f : \Omega \to \mathbb{C} \cup \{\infty\}$ is called meromorphic if it is holomorphic on Ω, except for isolated singularities that are poles. In other words, if $f(z) \neq \infty$ then f is holomorphic at z; if $f(z) = \infty$, then f has a pole at the isolated singularity z.

Theorem 0.5 (Argument Principle)**.** *Let $f : \Omega \to \mathbb{C} \cup \{\infty\}$ be a meromorphic function. Let γ be a simple closed curve on Ω. Suppose f has no zeros and poles on γ. Then we have*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = Z(f, \gamma) - P(f, \gamma)$$

where $Z(f, \gamma)$ and $P(f, \gamma)$ denote the number of zeros and poles (counting multiplicity) of f in the region bounded by the closed curve γ,

Remark.

- The proof follows from the Residue Theorems. The residues could be computed by writing $f(z) = g(z)(z - z_0)^n$ (resp. $f(z) = (z - z_0)^{-n}$) where n is the order of the zero (resp. pole) z_0 with g being holomorphic non-zero at z_0.
- The number of poles and zeros of a meromorphic function on the closed region bounded by a (closed) curve is always finite. This is because of the isolation of poles and zeros and the compactness of the (closed and bounded) region. (See Proposition 1.47 in the Lecture Note for a similar proof.)
- The integral on the left (including the multiple $\frac{1}{2\pi i}$) is in fact the so-called winding number[1]of the curve $f \circ \gamma$ at 0, which naively is the number of times the curve $f \circ \gamma$ (the image of the curve γ through f) loops around 0. It is also denoted by $\frac{1}{2\pi i} \Delta_\gamma(f(z))$ as in the textbook and lecture note.
- The Argument Principal shows that the image of a simple closed curve from a holomorphic map is not necessarily simple, that is, the image may have self-intersections.

Theorem 0.6 (Rouche's Theorem)**.** *Let $g, f : \Omega \to \mathbb{C}$ be holomorphic from a domain. Let γ be a simple closed curve on Ω. Suppose $|g(z)| < |f(z)|$ for all $z \in \gamma$. Then $f, g + f$ have the same number of zeros inside γ*

Remark.

- Its proof is an application of the Argument Principle.
- The Rouche's Theorem roughly says that you can modify a given holomophic function f without chaning the number of zeros inside certain region as long as the modification function g is strictly dominated by f on the boundary of the region.

[1]Let $\gamma : [s, t] \to \mathbb{C}$ be a (continuously differentiable) closed curve (not necessarily simple). (Recall that a simple curve is one that has no self-intersection except maybe at endpoints. So a (non-simple) closed curves may loop around itself many times). Let a be a point not on γ (so w either lies in the interior region bounded by the curve or the region exterior to the curve). Then the winding number of γ around a is *defined* to be $W_\gamma(a) := \frac{1}{2\pi i} \int_\gamma \frac{1}{z-a} dz$. Do you see why this definition really can capture the number of times a curve loops around a certain point?

Solutions

Throughout the solution, unless otherwise specified, C_R is understood to be the upper semi-circular arc with radius $R > 0$, oriented clockwise and centered at 0 . Moreover the integral $\int_a^b$ would always refer to integrating the contour $[a, b]$ on the real-axis where $a, b \in \mathbb{R}$. We also denote $[z_1, z_2]$ to be the straightline contour oriented from z_1 to z_2 where $z_1, z_2 \in \mathbb{C}$.

P.264-265

2. $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}.$

Solution. Let $f(z) = \frac{1}{(z^2+1)^2}$. Then f is holomorphic except at $\{i\}$ on the upper half plane, where i is an order-2 pole. By residue theorem on the upper half disks, fix $R > 0$ sufficiently large, we have

$$\int_{-R}^{R} f(z)dz + \int_{C_R} f(z)dz = 2\pi i \operatorname{Res}(f, i)$$

First, we compute $\operatorname{Res}(f, i)$. Since i is an order-2 pole, we define $g(z) = (z-i)^{-2} f(z)$ where g is holomorphic non-zero at i. Then $\operatorname{Res}(f, i) = g'(i) = (\frac{1}{(z+i)^2})'(i) = \frac{-i}{4}$.

Next note that for all $z \in C_R$, $|f(z)| = \left|\frac{1}{(z^2+1)^2}\right| \leq \frac{1}{(|z|^2-1)^2} = \frac{1}{(R^2-1)^2} =: M_R$. Therefore by the triangle inequality for integral, we have that $\left|\int_{C_R} f(z)dz\right| \leq \pi R M_R$ which converges to 0 as $R \to \infty$. Therefore putting $R \to \infty$, we have

$$\int_{-\infty}^{\infty} f(z)dz = 2\pi i \operatorname{Res}(f, i) = \frac{\pi}{2}$$

The result follows by the even-ness of f on the real-axis.

4. $\displaystyle\int_0^\infty \frac{x^2\,dx}{x^6+1} = \frac{\pi}{6}.$

Solution. Let $f(z) = \frac{z^2}{z^6+1}$. Then f is holomorphic except at $\{a_1 :=^{\pi/6}, a_2 := e^{\pi/2}, a_3 := e^{5\pi/6}\}$ on the upper half plane. By residue theorem on the upper half disks, fix $R > 0$ sufficiently large, we have

$$\int_{-R}^{R} f(z)dz + \int_{C_R} f(z)dz = 2\pi i \sum_{j=1}^{3} \operatorname{Res}(f, a_j)$$

First, we compute $\operatorname{Res}(f, a_j)$. Let α be a zero of $z^6 + 1$. Then α is a simple pole of $f(z)$. By the residule formula and L'Hospital Rule, we compute that

$$\operatorname{Res}(f, \alpha) = \lim_{z \to \alpha} (z-\alpha) f(z) = \lim_{z \to \alpha} \frac{(z-\alpha)z^2}{z^6+1} = \frac{\alpha^2}{6\alpha^5} = \frac{1}{6\alpha^3}$$

Note that $a_1^3 = i, a_2^3 = -i, a_3^3 = i$. Hence, we have $\sum_{j=1}^{3} \operatorname{Res}(f, a_j) = \frac{1}{6i}$.

Next note that for all $z \in C_R$, $|f(z)| = \left|\frac{z^2}{z^6+1}\right| \leq \frac{|z|^2}{|z|^6-1} = \frac{R^2}{R^6-1} =: M_R$. Therefore by the triangle inequality for integral, we have that $\left|\int_{C_R} f(z)dz\right| \leq \pi R M_R$ which converges to 0 as $R \to \infty$. Therefore putting $R \to \infty$, we have

$$\int_{-\infty}^{\infty} f(z)dz = 2\pi i \sum_{j=1}^{3} \operatorname{Res}(f, a_j) = \frac{\pi}{3}$$

The result follows by the even-ness of f on the real-axis.

9. Use a residue and the contour shown in Fig. 101, where $R > 1$, to establish the integration formula

$$\int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi}{3\sqrt{3}}.$$

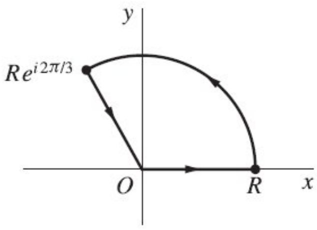

FIGURE 101

Solution. Let $\zeta_1 := e^{2\pi/3}, \zeta_2 := e^{4\pi/3}$. Let $f(z) = \frac{1}{z^3+1}$. Fix $R > 0$ sufficiently large.
First by a change of variable, observe that we have the following equality:

$$\int_{[R\zeta_1,0]} f(z)dz = -\zeta_1 \int_{[0,R]} f(w)dw$$

Second, denote the closed region (a circular sector) bounded by $[0,R],[0,R\zeta_1]$ by Ω_R. Note that f is holomorphic except at $a := e^{i\pi/3}$ on Ω_R.
Next, denote C_R the anti-clockwise circular arc from R to $\zeta_1 R$. Then by the residue theorem, we have

$$\left(\int_{[R\zeta_1,0]} + \int_{[0,R]} + \int_{C_R}\right) f(z)dz = 2\pi i \operatorname{Res}(f,a)$$

Since a is a simple pole, we compute that

$$\operatorname{Res}(f,a) = \lim_{z\to a}(z-a)f(z) = \lim_{z\to a}\frac{z-a}{z^3+1} = \frac{1}{3a^2} = \frac{1}{3\zeta_1}$$

Next it is easy to show that $\lim_{R\to\infty}\int_{C_R} f(z)dz = 0$ Therefore, with the formula in the beginning, we have

$$(1-\zeta_1)\int_0^\infty f(z)dz = 2\pi i \operatorname{Res}(f,a) = \frac{2\pi i}{3}\frac{1}{\zeta_1}$$

Note that $(\zeta_1(1-\zeta_1)) = \zeta_1 - \zeta_1^2 = \zeta_1 - \zeta_2 = \zeta_1 - \overline{\zeta_1} = 2i\operatorname{Im}(\zeta_1) = \sqrt{3}i$. Hence it follows that

$$\int_0^\infty f(z)dz = \frac{2\pi}{3\sqrt{3}}$$

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3. $\displaystyle\int_0^\infty \frac{\cos ax}{(x^2+b^2)^2}\,dx = \frac{\pi}{4b^3}(1+ab)e^{-ab} \quad (a>0, b>0).$

Solution. Let $g(z) = \frac{1}{(z^2+b^2)^2}$. Consider $f(z) = g(z)e^{iaz}$. Then f is holomorphic except at bi on the upper half plane. By residue theorem on the upper half disks, fix $R > 0$ sufficiently large, we have

$$\int_{-R}^{R} f(z)dz + \int_{C_R} f(z)dz = 2\pi i\,\mathrm{Res}(f, bi)$$

First, we compute $\mathrm{Res}(f, bi)$. Note that this is an order-2 pole. Take $f(z) = h(z)(z-bi)^2$ where h is holomorphic non-zero at bi. Then $\mathrm{Res}(f, bi) = h'(bi)$ by the residue formula.
Next note that for all $z \in C_R$, $|g(z)| = \left|\frac{1}{(z^2+b^2)^2}\right| \le \frac{1}{(|z|^2-|b|^2)^2} = \frac{1}{(R^2-|b|^2)^2} =: M_R$ which converges to 0 as $R \to \infty$. By the Jordan Lemma, $\int_{C_R} f(z)dz = \int_{C_R} g(z)e^{iaz}dz \to 0$ as $R \to \infty$. Therefore putting $R \to \infty$, we have

$$\int_{-\infty}^{\infty} f(z)dz = 2\pi i\,\mathrm{Res}(f, bi)$$

The result follows by considering the real part of g and its even-ness on the real-axis, plus the previously computed residue.
Remark. The result can follow without using the Jordan Lemma: note that $\left|e^{iaz}\right| \le 1$ for z on the upper half plane as $a > 0$. In this case, the triangle inequality on integrals still tells you $\int_{C_R} g(z)dz \to 0$ as $R \to \infty$.

5. $\displaystyle\int_{-\infty}^\infty \frac{x^3 \sin ax}{x^4+4}\,dx = \pi e^{-a}\cos a \quad (a>0).$

Solution. Let $g(z) = \frac{z^3}{z^4+4}$. Consider $f(z) = g(z)e^{iaz}$. Then f is holomorphic except at $\beta_1 := \sqrt{2}e^{i\pi/4, \beta_2 := \sqrt{2}e^{3\pi/4}}$ on the upper half plane. By residue theorem on the upper half disks, fix $R > 0$ sufficiently large, we have

$$\int_{-R}^{R} f(z)dz + \int_{C_R} f(z)dz = 2\pi i \sum_{j=1}^{2} \mathrm{Res}(f, \beta_j)$$

First, we compute $\mathrm{Res}(f, \beta_j)$. Let β be a zero of z^4+1. Then β is a simple pole of $f(z)$. By the residule formula and L'Hospital Rule, we compute that

$$\mathrm{Res}(f, \beta) = \lim_{z\to\beta}(z-\beta)f(z) = \lim_{z\to\beta}\frac{(z-\beta)z^3e^{iaz}}{z^4+1} = \frac{\beta^3e^{ia\beta}}{4\beta^3} = \frac{1}{4}e^{ia\beta}$$

Note that $e^{ia\beta_1} = e^{ia(1+i)} = e^{-a}e^{ia}$ and $e^{ia\beta_2} = e^{ia(-1+i)} = e^{-a}e^{-ia}$. Therefore $e^{ia\beta_1} + e^{ia\beta_2} = e^{-a}2\cos(a)$. Hence, $\sum_{j=1}^{2}\mathrm{Res}(f, \beta_j) = \frac{1}{2}e^{-a}\cos(a)$.
Next note that for all $z \in C_R$, $|g(z)| = \left|\frac{z^3}{z^4+4}\right| \le \frac{|z|^3}{|z|^4-4} = \frac{R^3}{R^4-4} =: M_R$ which converges to 0 as $R \to \infty$. By the Jordan Lemma, $\int_{C_R} f(z)dz = \int_{C_R} g(z)e^{iaz}dz \to 0$ as $R \to \infty$. Therefore putting $R \to \infty$, we have

$$\int_{-\infty}^{\infty} f(z)dz = 2\pi i \sum_{j=1}^{2}\mathrm{Res}(f, \beta_j) = i\pi e^{-a}\cos(a)$$

By considering the imaginary part of f, we have

$$\int_{-\infty}^{\infty}\frac{\sin(az)z^3}{z^4+4}dz = \int_{-\infty}^{\infty}\mathrm{Im}\left(\frac{e^{iaz}z^3}{z^4+4}\right) = \mathrm{Im}\left(\int_{-\infty}^{\infty}\frac{e^{iaz}z^3}{z^4+4}\right) = \mathrm{Im}\left(\int_{-\infty}^{\infty}f(z)dz\right) = \pi e^{-a}\cos(a)$$

8. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x \, dx}{x^2+4x+5}.$

Ans. $-\dfrac{\pi}{e}\sin 2.$

Solution. Let $g(z) = \frac{1}{z^2+1}$. $f(z) = e^{iz}g(z)$. Then f is holormophic on the upper half plane except at $z = i$. Fix $R > 0$ to be sufficiently large. Then by the Residue Theorem, we have

$$\int_{-R}^{R} f(z)dz + \int_{C_R} f(z)dz = 2\pi i \operatorname{Res}(f, i)$$

First, note the i is a simple pole. By the residue formula, we have $\operatorname{Res}(f, i) = \lim_{z\to i}(z - i)f(z) = e^{-1}/2i$.
Next, note that for all $z \in C_R$, $M_R := \sup_{z\in C_R}\{|g(z)|\} \to 0$. By the Jordan Lemma, we have $\int_{C_R} f(z)dz \to 0$ as $R \to \infty$.
Therefore, as $R \to \infty$, we have

$$\int_{-\infty}^{\infty} f(z)dz = 2\pi i \operatorname{Res}(f, i) = \pi e^{-1}$$

By considering the real part we have,

$$\int_{\infty}^{\infty} \frac{\cos z}{z^2+1} dz = \frac{\pi}{e}$$

The result follows by noting that with a change of variable, we have and taking limit afterards, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5} &= \lim_{R\to\infty} \int_{-R-2}^{R-2} \frac{\sin x}{x^2+4x+5} = \lim_{R\to\infty} \int_{-R}^{R} \frac{\sin(y-2)}{y^2+1} dy \\ &= \lim_{R\to\infty} \int_{-R}^{R} \frac{\sin(y)\cos(2) - \sin(2)\cos(y)}{y^2+1} dy = -\sin(2) \int_{-\infty}^{\infty} \frac{\cos y}{y^2+1} dy = -\frac{\pi}{e}\sin(2)\end{aligned}$$

12. Follow the steps below to evaluate the ***Fresnel integrals***, which are important in diffraction theory:

$$\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

(*a*) By integrating the function $\exp(iz^2)$ around the positively oriented boundary of the sector $0 \le r \le R, 0 \le \theta \le \pi/4$ (Fig. 106) and appealing to the Cauchy–Goursat theorem, show that

$$\int_0^R \cos(x^2)\,dx = \frac{1}{\sqrt{2}}\int_0^R e^{-r^2}\,dr - \operatorname{Re}\int_{C_R} e^{iz^2}\,dz$$

and

$$\int_0^R \sin(x^2)\,dx = \frac{1}{\sqrt{2}}\int_0^R e^{-r^2}\,dr - \operatorname{Im}\int_{C_R} e^{iz^2}\,dz,$$

where C_R is the arc $z = Re^{i\theta}$ $(0 \le \theta \le \pi/4)$.

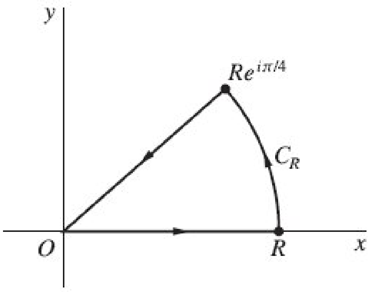

FIGURE 106

(*b*) Show that the value of the integral along the arc C_R in part (*a*) tends to zero as R tends to infinity by obtaining the inequality

$$\left|\int_{C_R} e^{iz^2}\,dz\right| \le \frac{R}{2}\int_0^{\pi/2} e^{-R^2\sin\phi}\,d\phi$$

and then referring to the form (2), Sec. 88, of Jordan's inequality.

(*c*) Use the results in parts (*a*) and (*b*), together with the known integration formula*

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2},$$

to complete the exercise.

Solution. Just follow the steps in question closely.

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1. Use the function $f(z) = (e^{iaz} - e^{ibz})/z^2$ and the indented contour in Fig. 108 (Sec. 89) to derive the integration formula

$$\int_0^\infty \frac{\cos(ax) - \cos(bx)}{x^2}\,dx = \frac{\pi}{2}(b-a) \qquad (a \geq 0, b \geq 0).$$

Then, with the aid of the trigonometric identity $1 - \cos(2x) = 2\sin^2 x$, point out how it follows that

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}.$$

Solution. Let $f(z) = \frac{e^{iaz}-e^{ibz}}{z^2}$. If $a = b$, then f is the zero function and so the integral is clear. Now suppose $a \neq b$. Then $f(z)$ is analytic except at 0. Therefore, for $R > \rho > 0$, we have by the Cauchy Goursat Theorem that

$$\left(\int_{-R}^{-\rho} + \int_{\rho}^{R} + \int_{-C_\rho} + \int_{C_R}\right) f(z)dz = 0$$

First note that for all $z \in C_R$, $|f(z)| = \left|\frac{e^{iaz}-e^{ibz}}{z^2}\right| \leq \frac{|e^{iaz}|+|e^{ibz}|}{|z|^2} \leq \frac{2}{R^2} =: M_R$. (Note that $\left|e^{iaz}\right| = |e^{-ay}| \leq 1$ with z on the upper half plane). Hence, by the triangle inequality for integrals, we have $\left|\int_{C_R} f(z)dz\right| \leq \pi R M_R$ which converges to 0 as $R \to \infty$.
Next, note that f is analytic on a deleted neighborhood of 0 and 0 is a simple pole of $f(z)$, which could be verified by considering the Taylor's expansion of $e^{iaz} - e^{ibz}$. Therefore the residue of $f(z)$ at 0 is given by

$$\operatorname{Res}(f,0) = \lim_{z\to 0} zf(z) = \lim_{z\to 0} \frac{e^{iaz}(1 - e^{ibz-iaz})}{z} = i(a-b) \neq 0$$

in which the fact that $\lim_{z\to 0}\frac{e^z-1}{z} = 1$ has been used. Hence, by the Indented Contour Approximation, we have

$$\lim_{\rho\to 0}\int_{C_\rho} f(z)dz = \pi i \operatorname{Res}(f,0) = (b-a)\pi$$

Combining the two result, we have

$$\int_{-\infty}^{\infty} f(z)dz = \lim_{\rho\to 0}\lim_{R\to\infty}\int_{-R}^{-\rho} f(z)dz + \int_{\rho}^{R} f(z)dz = \lim_{\rho\to 0}\lim_{R\to\infty}\int_{C_\rho} f(z)dz - \int_{C_R} f(z)dz = (b-a)\pi$$

The result follows by considering the real part of the above integral and further by the even-ness of it on the real line.

To show the last statement, we plug in $a = 1$ and $b = 2$ and note that $\cos x - \cos 2x = -2\sin^2(x/2) + 2\sin^2(x)$. A change of variable then gives the result.

2. Derive the integration formula

$$\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}$$

by integrating the function

$$f(z) = \frac{z^{-1/2}}{z^2+1} = \frac{e^{(-1/2)\log z}}{z^2+1} \qquad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

over the indented contour appearing in Fig. 109 (Sec. 90).

Solution. Following the examples in the previous section will do.

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1. $\displaystyle\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \frac{2\pi}{3}.$

2. $\displaystyle\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = \sqrt{2}\pi.$

3. $\displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta\, d\theta}{5-4\cos 2\theta} = \frac{3\pi}{8}.$

4. $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}} \quad (-1 < a < 1).$

Solution. All these follow from turning the integral in question to a contour integral using the substitution $z = e^{i\theta}$. We shall only show the first one.

1. Let $z = z(\theta) = e^{i\theta}$ where $\theta \in [0, 2\pi]$. Then $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-z^{-1}}{2i}$ and $dz = ie^{i\theta}d\theta = izd\theta$. Hence, we could compute that

$$\int_0^{2\pi} \frac{1}{5+4\sin\theta}d\theta = \int_C \frac{1}{iz(5-2iz+2iz^{-1})}dz = \int_C \frac{1}{2z^2+5iz-2}dz = \int_C \frac{1}{(2z+i)(z+2i)}dz$$

   where C denotes the unit circle (oriented anti-clockwise). Let $f(z) = \frac{1/2}{z+2i}$ and $w = -i/2$. Then f is holomorphic inside and on C while w is in the interior of the region bounded by C. By the Cauchy-Integral Formula (of course you could use the Residue Theorem as well), we conclude that

$$\int_0^{2\pi} \frac{1}{5+4\sin\theta}d\theta = \int_C \frac{1}{(2z+i)(z+2i)}dz = 2\pi i f(w) = 2\pi i\frac{1}{3i} = \frac{2\pi}{3}$$

2. It is similar to Part 1, but more troublesome when computing the residues. You may use the L'Hospital rule to simplify the computation

3. Similar to Part 1.

4. Similar to Part 1.

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1. Let C denote the unit circle $|z| = 1$, described in the positive sense. Use the theorem in Sec. 93 to determine the value of $\Delta_C \arg f(z)$ when
(a) $f(z) = z^2$; (b) $f(z) = 1/z^2$; (c) $f(z) = (2z-1)^7/z^3$.
Ans. (a) 4π; (b) -4π; (c) 8π.

Solution. (a) Note that $f(z) = z^2$ is meromorphic (in fact holomorphic) on $\mathbb{C}$ and f is non-zero holomophic on $C \subset \mathbb{C}$. Hence by Argument Principal, the winding number of $f(C)$ at 0 is given by

$$\frac{1}{2\pi i}\Delta_C \arg f(z) = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N(f,C) - P(f,C) = 2 - 0 = 2$$

Hence we have $\Delta_C \arg f(z) = 4\pi$

(b) Note that $f(z) = 1/z^2$ is meromorphic on $\mathbb{C}$ and f is non-zero holomophic on $C \subset \mathbb{C}$. Hence by Argument Principal, the winding number of $f(C)$ at 0 is given by

$$\frac{1}{2\pi i}\Delta_C \arg f(z) = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N(f,C) - P(f,C) = 0 - 2 = -2$$

Hence we have $\Delta_C \arg f(z) = -4\pi$

(c) Note that $f(z) = (2z-1)^7/z^3$ is meromorphic on $\mathbb{C}$ and f is non-zero holomophic on $C \subset \mathbb{C}$. Hence by Argument Principal, the winding number of $f(C)$ at 0 is given by

$$\frac{1}{2\pi i}\Delta_C \arg f(z) = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N(f,C) - P(f,C) = 7 - 3 = 4$$

which follows from that $1/2$ and 0 are order-7 zero and order-3 pole inside the region bounded by C respectively. Hence we have $\Delta_C \arg f(z) = 8\pi$

2. Let f be a function which is analytic inside and on a positively oriented simple closed contour C, and suppose that $f(z)$ is never zero on C. Let the image of C under the transformation $w = f(z)$ be the closed contour Γ shown in Fig. 114. Determine the value of $\Delta_C \arg f(z)$ from that figure; and, with the aid of the theorem in Sec. 93, determine the number of zeros, counting multiplicities, of f interior to C.
Ans. 6π; 3.

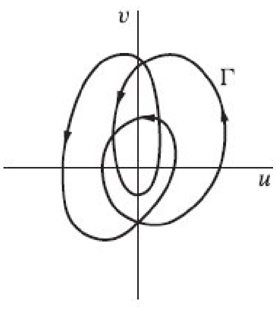

FIGURE 114

Solution. To obtain the winding number from a picture, you fix a point and draw a line from the origin to that point. Then you travel the point along the curve in the assigned direction and count the number of cycles that line connecting the origin and the initial point has traveled. That number is the winding number of the curve at 0.

6. Determine the number of zeros, counting multiplicities, of the polynomial

(*a*) $z^6 - 5z^4 + z^3 - 2z$; (*b*) $2z^4 - 2z^3 + 2z^2 - 2z + 9$; (*c*) $z^7 - 4z^3 + z - 1$.

inside the circle $|z| = 1$.

Ans. (*a*) 4 ; (*b*) 0; (*c*) 3.

Solution. (a). Apply Rouche's Theorem with $f(z) = -5z^4$ and $g(z) = z^6 + z^3 - 2z$ on $|z| = 1$

(b). Apply Rouche's Theorem with $f(z) = -5z^4$ and $g(z) = z^6 + z^3 - 2z$ on $|z| = 1$

(c). Apply Rouche's Theorem with $f(z) = -4z^3$ and $g(z) = z^7 + z - 1$ on $|z| = 1$

8. Determine the number of roots, counting multiplicities, of the equation

$$2z^5 - 6z^2 + z + 1 = 0$$

in the annulus $1 \leq |z| < 2$.

Ans. 3.

Solution. Apply Rouche's Theorem on $|z| = 2$ and $|z| = 1$ with suitable functions repsectively. Then the difference is the answer.