MATH 2230A - HW 10 - Solutions

The computation techniques demonstrated in this Homework is EXTREMEMLY important.
Please make sure you are familiar with the techniques
Full solutions at P.84-85 Q5, P.254 Q5, Q6
Commonly missed steps in Purple and common mistakes at the back

Below are some facts useful to this homework (especially for questions on P.84-85).

Theorem 0.1 (Isolation of Zeros). *Let $f : \Omega \to \mathbb{C}$ be holomorphic on a domain. Suppose f has a zero at $a \in \Omega$, that is, $f(a) = 0$. Then there exists a neighborhood $B(a, r)$ of a such that either $f(z) = 0$ for all $z \in B(a, r)$ or $f(z) \neq 0$ for all $z \in B(a, r)\backslash\{z\}$.*

Remark. This is easily proven from Taylor Series. In fact by the *connectedness* of domain, one can strength the result to that either f is constantly 0 on Ω or f only can have isolated zeros on Ω

Theorem 0.2 (Coincidence Principle for holomorphic functions). *Let $f : \Omega \to \mathbb{C}$ be holomorphic on a (open connected) domain. Suppose $f = 0$ on D where $D \subset \Omega$ is a subset containing an accumulation point*[1]*. Then $f = 0$ on Ω.*

Remark. The result basically follows from the isolation of zeros for holomorphic functions. The assumption that Ω is connected is important as seen from the remark in the Isolation of Zeros.

Corollary 0.3. *Let $f, g : \Omega \to \mathbb{C}$ be holomorphic on a domain. Suppose $f = g$ on some sub-domain or a line in Ω. Then $f = g$.*

Remark. This is because sub-domains (open-connected subsets) or lines have accumulation points.

Theorem 0.4 (Reflection Principle). *Let $\Omega \subset \mathbb{C}$ be a domain that is symmetric along the real-axis, that is, for all $z \in \mathbb{C}$ we have $z \in \Omega$ if and only if $\overline{z} \in \Omega$. Let $f : \Omega \to \mathbb{C}$ be holomorphic. Suppose that*

1. *$\ell := \Omega \cap \mathbb{R}$ is non-empty and lies in the interior of Ω*
2. *f is real on ℓ.*

Then we have $\overline{f(z)} = f(\overline{z})$.

Remark. This basically follows from the co-incidence principle. We first observe that the function $g : \Omega \to \mathbb{C}$ defined by $g(z) = \overline{f(\overline{z})}$ is holomorphic (by possibly considering the Cauchy Riemann Equations). Then we show that $f = g$ on ℓ, which is not difficult. Since ℓ, which is a line, contains an accumulation point, the result follows by extending the equality to the whole domain Ω.

Please refer to HW9 Solutions for Theorems that are related to poles and residues.

[1]Let $D \subset \mathbb{C}$ be a subset. Then we call $z_0 \in D$ an accumulation point if for all neighborhood $U := B(z_0, r)$ of z_0 where $r > 0$, we have $U \cap D\backslash\{z_0\} \neq \phi$. In other words, we can find a sequence (z_n) in D with every element not being z_0 such that $z_n \to z_0$. Roughly speaking, an accumulation point in D are those that are not "isolated" from other members in D.

P.84-85

2. Starting with the function

$$f_1(z) = \sqrt{r}e^{i\theta/2} \qquad (r > 0, 0 < \theta < \pi)$$

and referring to Example 2, Sec. 24, point out why

$$f_2(z) = \sqrt{r}e^{i\theta/2} \qquad \left(r > 0, \frac{\pi}{2} < \theta < 2\pi\right)$$

is an analytic continuation of f_1 across the negative real axis into the lower half plane. Then show that the function

$$f_3(z) = \sqrt{r}e^{i\theta/2} \qquad \left(r > 0, \pi < \theta < \frac{5\pi}{2}\right)$$

is an analytic continuation of f_2 across the positive real axis into the first quadrant but that $f_3(z) = -f_1(z)$ there.

Solution. Please follow the examples in the textbook.

5. Show that if the condition that $f(x)$ is real in the reflection principle (Sec. 29) is replaced by the condition that $f(x)$ is pure imaginary, then equation (1) in the statement of the principle is changed to

$$\overline{f(z)} = -f(\overline{z}).$$

Solution. We need to show that if $\Omega \subset \mathbb{C}$ is a domain that is symmetric along the real-axis, that is, for all $z \in \mathbb{C}$ we have $z \in \Omega$ if and only if $\overline{z} \in \Omega$, and $f : \Omega \to \mathbb{C}$ is holomorphic on Ω such that

1. $\ell := \Omega \cap \mathbb{R}$ is non-empty and lies in the interior of Ω
2. f is *purely imaginary* on ℓ.

Then we have $\overline{f(z)} = -f(\overline{z})$.

Method 1: Without using the Reflection Principle

Let $g(z) = -\overline{f(\overline{z})}$. Then it suffices to show that $h(z) := f(z) - g(z)$ is constantly 0.
First, we claim that g is holomorphic on Ω. Denote u_g, v_g the real and imaginary part of g respectively. Then $u_g(x,y) = -u_f(x,-y)$ and $v_g(x,y) = v_f(x,-y)$ for all $z = x + iy \in \Omega$. Hence, we further have

$$\partial_x u_g(x,y) = -\partial_x u_f(x,-y) \qquad \partial_y u_g(x,y) = \partial_y u_f(x,-y)$$
$$\partial_x v_g(x,y) = \partial_x v_f(x,-y) \qquad \partial_y v_g(x,y) = \partial_y - v_f(x,-y)$$

Hence, by the analyticity of f, we have

$$\partial_x u_g(x,y) = -\partial_x u_f(x,-y) = -\partial_y v_f(x,-y) = \partial_y v_g(x,y)$$
$$\partial_y u_g(x,y) = \partial_y u_f(x,-y) = -\partial_x v_f(x,-y) = -\partial_x v_g(x,y)$$

Therefore, g satisfies the CR equations throughout Ω with continuously differentiable partial derivatives. Hence, g is holomophric on Ω. Therefore $h := f - g$ is holomorphic on Ω.
Now let $z \in \ell \subset \mathbb{R}$, then $g(z) = -\overline{f(\overline{z})} = -\overline{f(z)} = f(z)$ by assumption. Hence $h(z) = f(z) - g(z) = 0$ for all $z \in \ell$. Since h is holomorphic (as the difference of holomorphic functions) and ℓ is a line (which contains an accumulation point), we can extend the equality to the whole domain Ω by the coincidence principle.

Method 2: Using the Reflection Principle

Define $g(z) := if(z)$. Then g is holomorphic as it is just a scalar multiple of the holomrophic function f and g is real on ℓ. Hence by the Reflection Principle, we have that $if(z) = g(z) = \overline{g(\overline{z})} = \overline{if(\overline{z})} = -i\overline{f(\overline{z})}$ for all $z \in \Omega$. It follows that $f(z) = -\overline{f(\overline{z})} \Leftrightarrow \overline{f(z)} = -f(\overline{z})$ for all $z \in \Omega$.

Remark. Please write clearly every time the domain of functions you are referring to, especially for this question.

P.246

1. In each case, show that any singular point of the function is a pole. Determine the order m of each pole, and find the corresponding residue B.

(a) $\dfrac{z+1}{z^2+9}$; (b) $\dfrac{z^2+2}{z-1}$; (c) $\left(\dfrac{z}{2z+1}\right)^3$; (d) $\dfrac{e^z}{z^2+\pi^2}$.

Ans. (a) $m=1, B=\dfrac{3\pm i}{6}$; (b) $m=1, B=3$; (c) $m=3, B=-\dfrac{3}{16}$;

(d) $m=1, B=\pm\dfrac{i}{2\pi}$.

Solution. 1. $f(z) = \frac{z+1}{z^2+9}$. f has isolated singularities at $\pm 3i$. Note $f(z) = \frac{1}{z\pm 3i}g(z)$ where $g(z) := \frac{z+1}{z\mp 3i}$ is holomorphic non-zero at $z=\mp 3i$. This shows that f has a simple pole at $\pm 3i$.
We compute the residues by: $\mathrm{Res}(f,\pm 3i) = \lim_{z\to\pm 3i}(z\mp 3i)f(z) = \frac{\mp 3i+1}{\mp 3i\mp 3i} = \frac{3+\pm i}{6}$.

2. $f(z) = \frac{z^2+2}{z-1}$. f has isolated singularities at . Note $f(z) = \frac{1}{z-1}g(z)$ where $g(z) := z^2+2$ is holomorphic non-zero at $z=1$. This shows that f has a simple pole at 1.
We compute the residues by: $\mathrm{Res}(f,1) = \lim_{z\to 1}(z-1)f(z) = g(1) = 3$.

3. $f(z) = (\frac{z}{2z+1})^3$. f has isolated singularities at $-1/2$. Note $f(z) = \frac{1}{(z+1/2)^3}g(z)$ where $g(z) := (z/2)^3$ is holomorphic non-zero at $z=-1/2$. This shows that f has a pole of order 3 at 1.
We compute the residues by: $\mathrm{Res}(f,-1/2) = \frac{d^2}{dz^2}\Big|_{z=-1/2}\frac{(z+1/2)^3 f(z)}{2!} = \frac{g''(-1/2)}{2} = \frac{-3}{16}$.

4. $f(z) = \frac{e^z}{z^2+\pi^2}$. f has isolated singularities at $\pm\pi i$. Note $f(z) = \frac{1}{z\pm\pi i}g(z)$ where $g(z) := \frac{e^z}{z\mp\pi i}$ is holomorphic non-zero at $z=\mp\pi i$. This shows that f has a simple poles at $\mp\pi i$.
We compute the residues by: $\mathrm{Res}(f,\mp\pi i) = \lim_{z\to\mp\pi i}(z\pm\pi i)f(z) = g(\mp\pi i) = \frac{e^{\mp\pi}}{\mp 2\pi i} = \mp\frac{i}{2\pi}$.

P.254

5. Let C denote the positively oriented circle $|z|=2$ and evaluate the integral

(a) $\displaystyle\int_C \tan z\,dz$; (b) $\displaystyle\int_C \frac{dz}{\sinh 2z}$.

Ans. (a) $-4\pi i$; (b) $-\pi i$.

Solution. (a) Let Ω be the closed region bounded by C. Let $f(z) = \tan z$. Then $\tan z = \frac{\sin z}{\cos z}$. Note that

$$\cos z = 0 \iff e^{zi} + e^{-zi} = 0 \iff e^{2zi} = -1 \iff 2zi \in log(-1)$$
$$\iff 2zi = (2n+1)\pi i, \exists n\in\mathbb{Z} \iff z = \frac{(2n+1)\pi}{2}, \exists n\in\mathbb{Z}$$

Hence $f(z)$ is not analytic except at $z_n := \frac{(2n+1)\pi}{2}$ for some $n\in\mathbb{Z}$. It is easy to see that these singularities of f are isolated.
Note that $\pm\frac{\pi}{2}\in\Omega^o$ and f is holomorphic on Ω (which is simply connected) except at these 2 points. Hence, by Residue Theorem, we have

$$\int_C \tan z dz = 2\pi i(\mathrm{Res}(f,\pi/2) + \mathrm{Res}(f,-\pi/2))$$

Next we check that these singularities are poles and compute their orders. Note that $\cos(z_n) = 0$ where z_n were defined to be the singularities, but $\sin(z_n) = (\cos(z))'(z_n) = \pm 1 \neq 0$. Hence all these isolated singularities are zeros of order 1 for $\cos z$. This implies for all $n\in\mathbb{Z}$, there exists $\phi_n(z)$ holomorphic non-zero at z_n and locally(in a neighborhood of z_n), we have that

$\cos z = (z - z_n)\phi_n(z)$. Therefore for all $n \in \mathbb{Z}$, locally we have $f(z) = \tan(z) = \frac{\sin z}{(z-z_n)\phi_n(z)}$. Note that $\sin z, \phi_n(z)$ are all holomorphic non-zero at z_n. Hence, f has simple poles at z_n for all $n \in \mathbb{Z}$.
Since all these isolated singularities are simple poles, we compute the residues as follows:

$$\text{Res}(f, \pi/2) = \lim_{z\to\pi/2}(z - \pi/2)\frac{\sin z}{\cos z} = \lim_{w\to 0} w\frac{\sin(w+\pi/2)}{\cos(w+\pi/2)} = \lim_{w\to 0}\frac{w\cos w}{-\sin w} = -1$$

and

$$\text{Res}(f, -\pi/2) = \lim_{z\to-\pi/2}(z + \pi/2)\frac{\sin z}{\cos z} = \lim_{w\to 0} w\frac{\sin(w-\pi/2)}{\cos(w-\pi/2)} = \lim_{w\to 0}\frac{-w\cos w}{\sin w} = -1$$

We have used the fact that $\lim_{w\to 0}\frac{\sin w}{w} = 1$. Therefore, we have $\int_C \tan z dz = 2\pi i(-1-1) = -4\pi i$.

(b) Let Ω be the closed region bounded by C. Let $f(z) = \frac{1}{\sinh 2z}$. Note that

$$\sinh 2z = 0 \iff e^{2z} - e^{-2z} = 0 \iff e^{4z} = 1 \iff 4z \in log(1)$$
$$\iff 4z = 2n\pi i, \exists n \in \mathbb{Z} \iff z = \frac{n\pi}{2}i, \exists n \in \mathbb{Z}$$

Hence $f(z)$ is not analytic except at $z_n := \frac{n\pi}{2}i$ for some $n \in \mathbb{Z}$. It is easy to see that these singularities of f are isolated. Note that $0, \pm\pi/2 \in \Omega^0$ and f is holomorphic on Ω except at these 2 points. Hence, by Residue Theorem, we have

$$\int_C f(z)dz = 2\pi i(\text{Res}(f,0) + \text{Res}(f, i\pi/2) + \text{Res}(f, -i\pi/2))$$

Next, we show that these singularities are poles and compute their orders. Note that for all $n \in \mathbb{Z}$, $\sinh(2z_n) = 0$, but $(\sinh 2z)'(z_n) = 2\cosh(2z_n) = 2\cosh(n\pi i) = 2\cos(n\pi) \neq 0$. Hence all these isolated singularities are zeros of order 1 for $\sinh 2z$, which implies for all $n \in \mathbb{Z}$, there exists $\phi_n(z)$ holomorphic non-zero at z_n and locally(in a neighborhood of z_n), we have$\sinh 2z = (z-z_n)\phi_n(z)$. Therefore for all $n \in \mathbb{Z}$, locally we have $f(z) = \frac{1}{\sinh(z)} = \frac{1}{(z-z_n)\phi_n(z)}$ where $\phi_n(z)$ are all holomorphic non-zero at z_n. Hence, f has simple poles at z_n for all $n \in \mathbb{Z}$.
Since all these isolated singularities are simple poles, we compute the residues as follows: Let a be an isolated singularity and let $g_a(z)$ be holomorphic non-zero at a such that locally $\frac{g_a(z)}{z-a} = f(z)$. Let $h(z) := \sinh(z)$, then $(z)g_a(z) = z - a$ locally at 0. By considering Laurent(Taylor) Series at a, we have

$$\begin{aligned}\sinh(2z)g_a(z) = h(z)g_a(z) &\left(\sum_{i=0}\frac{h^{(i)(a)}(z-a)^j}{j!}\right)\left(\sum_{j=0}\frac{g_a^{(j)}(a)(z-a)^j}{j!}\right)\\ &= \left(h(a) + h'(a)(z-a) + \frac{h''(a)}{2}(z-a)^2 + \dots\right)\left(g_a(a) + g_a'(a)z + \frac{g_a''(a)}{2}(z-a)^2 + \dots\right)\\ &= z - a\end{aligned}$$

By comparing like terms, as $\sinh(a) = h(a) = 0$ we have $h'(a)g_a(a) = 1$. Therefore, $g_a(a) = 1/h'(a)$. Since $h'(a) = (\sinh(2z))'(a) = 2\cosh(2a)$, we have $h'(0) = 2, h'(i\pi/2) = 2\cos(\pi) = -2, h'(-i\pi/2) = 2\cos(-\pi) = -2$. Hence, $g_a(a) = 1/2, -1/2, -1/2$ at $a = 0, i\pi/2, -i\pi/2$ respectively. Lastly but not least, we have for all singularities a of f,

$$\text{Res}(f, a) = \lim_{z\to a}\frac{z-a}{\sinh 2z} = \lim_{z\to a}\frac{z-a}{h(z)} = \lim_{z\to a} g_a(z) = g_a(a)$$

Hence, we have

$$\begin{aligned}\int_C f(z)dz &= 2\pi i(\text{Res}(f,0) + \text{Res}(f, i\pi/2) + \text{Res}(f, -i\pi/2))\\ &= 2\pi i \sum_{a=0, i\pi/2, -i\pi/2}(g_a(a)) = 2\pi i(1/2 - 1/2 - 1/2) = -\pi i\end{aligned}$$

Remark. For part b, the comparing like term technique is literally the same as the long division. Both follows from the convergence of Cauchy Products of Taylor's Series.

6. Let C_N denote the positively oriented boundary of the square whose edges lie along the lines

$$x = \pm\left(N + \frac{1}{2}\right)\pi \quad \text{and} \quad y = \pm\left(N + \frac{1}{2}\right)\pi,$$

where N is a positive integer. Show that

$$\int_{C_N} \frac{dz}{z^2 \sin z} = 2\pi i\left[\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}\right].$$

Then, using the fact that the value of this integral tends to zero as N tends to infinity (Exercise 8, Sec. 47), point out how it follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

Solution. Let $h(z) = z^2 \sin z$. Note that $\sin z = 0$ if and only if $z = n\pi$ for some $n \in \mathbb{Z}$. Furthermore, if $\sin(z) = 0$, $(\sin)'(z) = \cos(z) \neq 0$, so all zeros of $\sin z$ are of order 1.Therefore it is clear that h has an order 3 zero at 0 while it has order 1 zeros at $n\pi$ for all $0 \neq n \in \mathbb{Z}$. Hence $f(z) := 1/h(z) = \frac{1}{z^2 \sin z}$ has an order-3 pole at 0 and simple poles at $n\pi$ where $0 \neq n \in \mathbb{Z}$.
If $n \neq 0$, then the residue at $n\pi$ of f is given by

$$\text{Res}(f, n\pi) = \lim_{z\to n\pi}(z - n\pi)f(z) = \lim_{w\to 0} wf(w + n\pi) = \lim_{w\to 0} \frac{1}{(w + n\pi)^2} \frac{w}{\sin(w + n\pi)} = \frac{(-1)^n}{n^2\pi^2}$$

If $n = 0$, then $f(z) = g_0(z)/z^3$ locally at 0 for some g_0 holomorhpic non-zero at 0. Hence $z = g_0(z)\sin(z)$ locally at 0. By considering Taylor Series at 0 we have

$$z = \left(g_0(0) + g_0'(0)z + \frac{g_0''(0)z^2}{2} + \ldots\right)(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots)$$

Hence, by comparing like terms (for z, z^3), we have the system of equations:

$$g_0(0) = 1$$
$$\frac{g_0''(0)}{2} - \frac{g_0(0)}{3!} = 0$$

So, $g_0''(0) = 1/3$. Hence, we can compute the residue of f at 0 by

$$\text{Res}(f, 0) = \frac{d^2}{dz^2}\bigg|_{z=0} \frac{z^3 f(z)}{2!} = \frac{d^2}{dz^2}\bigg|_{z=0} \frac{g_0(z)}{2!} = \frac{g''(0)}{2} = \frac{1}{6}$$

Fix $N \in \mathbb{N}$. Let Ω_N be the closed region bounded by C_N. Observe that $n\pi \in \Omega_N^o$ if and only if $|n| \leq N$ and f is holomorphic on Ω except only at $n\pi \in \Omega_N^o$. Hence, by the Residue Theorem, we have

$$\frac{1}{2\pi i}\int_{C_N} \frac{dz}{z^2 \sin z} = \frac{1}{2\pi i}\int_{C_N} f(z)dz = \sum_{n=-N}^{N} \text{Res}(f, n\pi) = \frac{1}{6} + \sum_{n=-N, n\neq 0}^{N} \frac{(-1)^n}{n^2\pi^2} = \frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}$$

We then finally have

$$\int_{C_N} \frac{dz}{z^2 \sin z} = 2\pi i\left[\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}\right]$$

As it is given that the integral tends to 0 as $N \to \infty$, we can conclude by simply taking limit for the modulus of the right-hand side above that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

Remark. In fact the the integral tends to zero follows from the following: first we note the inequality

$$\left| \int_{C_N} \frac{dz}{z^2 \sin z} \right| \leq \int_{C_N} \left| \frac{1}{z^2 \sin z} \right| |dz|$$

Then we observe $\sin z = \sin(x + iy) = \sin x \cosh(y) + i \sinh(y) \cos(x)$ and we have $|\sin z|^2 = \sin^2(x) + \sinh^2(y)$ So, when $x = \pm(N + 1/2)\pi$, $N \in \mathbb{N}$, we have $|\sin z| \geq |\sin x| = 1$. When $y = \pm(N + 1/2)\pi$, we have $|\sin z| \geq |\sinh y| = |\sinh(N + 1/2)\pi| \geq \sinh(\pi/2)$ (as sinh is increasing on positive real-axis). Combining these observations, $|\sin z| \geq A$ where $A := \max\{1, \sinh(\pi/2)\}$ for all z on C_N is independent of N. Together with the fact that $|z| \geq (N + 1/2)\pi$ on C_N, we can further approximate the above inequality by

$$\left| \int_{C_N} \frac{dz}{z^2 \sin z} \right| \leq \int_{C_N} \left| \frac{1}{z^2 \sin z} \right| |dz| \leq \frac{|C_N|}{A(N + 1/2)^2 \pi^2} = \frac{(4N + 2)\pi}{A(N + 1/2)^2 \pi^2}$$

It the follows that the integral tends to 0 as $N \to \infty$

Remark. With the fact that $\sum_{n=0}^{\infty} \frac{1}{n}$ converges absolutely (how?), and hence its unconditional convergence, we could deduce from this question the solution to the famous Basel problem:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$