MATH 2230A - HW 1
Due Date: 18 Sep 2020, 23:59
(Please submit assignments to Blackboard and follow the instructions there.)

Problems: P.5 Q2; P.13-14 Q3, 4, 5, 6; P.23-24 Q1, 2, 4, 5, 7, 9 (11 Questions in total)

Textbook: Brown JW, Churchill RV(2014). Complex Variables and Applications, nineth edition, McGraw-Hill Education

We type here all the required problems *for your convenience only.* The presentation of the problems here may be different from the original one but the respective solution should be unaffected.

1 (P.5 Q2)**.** Let $z \in \mathbb{C}$. Prove the following statements.

a) $\operatorname{Re}(iz) = -\operatorname{Im}(z)$ b) $\operatorname{Im}(iz) = \operatorname{Re}(z)$

2 (P.13-14 Q3)**.** Let $z_i \in \mathbb{C}$ for $i = 1, 2, 3, 4$. Suppose $|z_3| \neq |z_4|$. Show that

$$\frac{\operatorname{Re}(z_1 + z_2)}{|z_3 + z_4|} \leq \frac{|z_1| + |z_2|}{||z_3| - |z_4||}$$

3 (P.13-14 Q4)**.** Let $z \in \mathbb{C}$. Show that $\sqrt{2}|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z|$

4 (P.13-14 Q5)**.** For each of the following, sketch the respective set of points determined by the equation.

a) $|z - 1 + i| = 1$ b) $|z + i| \leq 3$ c) $|z - 4i| \geq 4$

where $z \in \mathbb{C}$

5 (P.13-14 Q6)**.** Using the fact that $|z_1 - z_2|$ is the distance between points z_1, z_2, give a (geometric) argument of the following statement:

$|z - 1| = |z + i|$ represents the line through the origin whose slope is -1

6 (P.23-24 Q1)**.** Let $z \in \mathbb{C}$. Find the principal argument $\operatorname{Arg}(z)$ when

a) $z = \frac{-2}{1+\sqrt{3}i}$ b) $z = (\sqrt{3} - i)^6$

7 (P.23-24 Q2)**.** Let $\theta \in \mathbb{R}$. Prove the following identities:

a) $|e^{i\theta}| = 1$ b) $\overline{e^{i\theta}} = e^{-i\theta}$

8 (P.23-24 Q4)**.** It is given that $|e^{i\theta} - 1|$ is the distance between points $e^{i\theta}$ and 1.

(i). Find a candidate $\theta \in [0, 2\pi)$ such that $|e^{i\theta} - 1| = 2$

(ii). Verify your choice with a (geometric) argument.

9 (P.23-24 Q5)**.** By writing the *individual* factors on the *left* in *polar form*, performing the needed operations and finally changing back to planar form, show that

a) $i(1 - \sqrt{3}i)(\sqrt{3} + i) = 2(1 + \sqrt{3}i)$ b) $\frac{5i}{(2+i)} = 1 + 2i$

c) $(\sqrt{3} + i)^6 = -64$ d) $(1 + \sqrt{3}i)^{-10} = 2^{-11}(-1 + \sqrt{3}i)$

10 (P.23-24 Q7)**.** Let $0 \neq z \in \mathbb{C}$. Write $z = re^{i\theta}$ for some $\theta \in \mathbb{R}$ and $r > 0$. Let $m \in \mathbb{N}$. It is given that

$$z^m = r^m e^{im\theta} \text{ and } z^{-1} = \frac{1}{r}e^{i(-\theta)}$$

(i). Show that $(z^m)^{-1} = (z^{-1})^m$.

(ii). Hence, show that the following are equivalent, that is, (a) if and only if (b):

(a) For all negative integers n, non-zero $z \in \mathbb{C}$, and $m := -n$, we have $z^n = (z^{-1})^m$.

(b) For all negative integers n, non-zero $z \in \mathbb{C}$, and $m := -n$, we have $z^n = (z^m)^{-1}$.

11 (P.23-24 Q9)**.** This exercise guides you to prove the Lagranage's trigonometric identity.

(i). Prove the following identity for $1 \neq z \in \mathbb{C}$ and $n \in \mathbb{N}$.

$$1 + z + z^2 + \ldots + z^n = \frac{1 - z^{n+1}}{1 - z}$$

(ii). Hence, prove the **Lagrange's trigonometric identity**, that is, for $\theta \in (0, 2\pi)$ and $n \in \mathbb{N}$, we have

$$1 + \cos\theta + \cos 2\theta + \ldots + \cos n\theta = \frac{1}{2} + \frac{\sin\left[(2n+1)\theta/2\right]}{2\sin\theta/2}$$