Side-pairing transformations, Elliptic and Parabolic cycles, Poincaré's Theorem Cf.

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MATH4900E Group 3

Content

1. Side-pairing transformations

Recall

Defintion (hyperbolic polygon)

A hyperbolic polygon is a **closed convex set** in the hyperbolic plane that can be expressed as the intersection of a locally finite collection of **closed half-planes**.

Defintion (Convex set)

A subset X of the hyperbolic plane is convex if for each pair of points x and y in X, the closed hyperbolic line segment l_{xy} joining x to y is contained in X

Recall

Definition (Discrete group)

A subgroup G $\subset$ $SL(2,\mathbb{R})$ is a discrete group if G has no accumulation points in $SL(2,\mathbb{R})$.

Accumulation points

x is said to be an accumulation point in A if every open set containing x contains at least one other point from A.

Recall

Definition (Fuchsian group)

It is a discrete subgroup of either $\text{Möb}(\mathbb{H})$ or $\text{Möb}(\mathbb{D})$

Definition (Dirichlet polygon)

Each Fuchsian group possesses a fundamental domain. The purpose of the following slides is to give a method for constructing a fundamental domain for a given Fuchsian group. The fundamental domain that we construct is called a **Dirichlet polygon.**

Side-pairing transformations

Definition

Let D be a hyperbolic polygon. A side s $\in \mathbb{H}$ of D is an edge of D in $\mathbb{H}$ equipped with an orientation.

That is, a side of D is an edge which starts at one vertex and ends at another.

Let Γ be a Fuchsian group and let D(p) be a Dirichlet polygon for Γ.
We assume that D(p) has finitely many sides. Let s be a side of D.
Suppose that for some $\gamma \in \Gamma \setminus \{Id\}$
we have that γ(s) is also a side of D(p).
Note that $\gamma^{-1} \in \Gamma \setminus \{Id\}$ maps the side γ(s) back to the side s.

Then, the sides s and γ(s) are paired and call γ a **side pairing transformation**.

Side-pairing transformations

Remark

It is possible that s and $\gamma(s)$ are the same side, with opposing orientations.

Then, s is paired with itself.

Side-pairing transformations

Recall (Perpendicualr bisector)

Let $z_1, z_2 \in \mathbb{H}$ Recall that $[z_1, z_2]$ is the segment of the unique geodeic from z_1 to z_2.

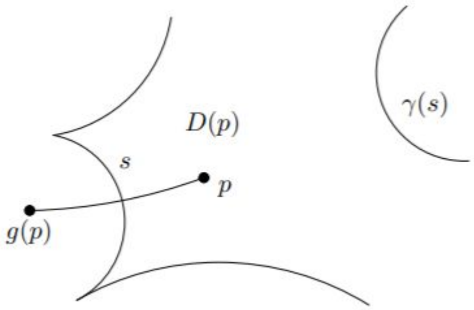
The perpendicular bisector of $[z_1, z_2]$ is defined to be the unique geodesic perpendicular to $[z_1, z_2]$ that passes through the midpoint of $[z_1, z_2]$.

Side-pairing transformations

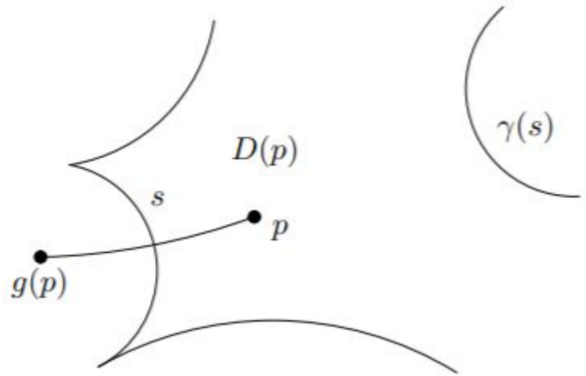
ways to find a side-pairing transformation associated to it

Let s be a side of a Dirichlet polygon D(p), then we can see that s is contained in the perpendicular bisector of the segment [p, g(p)], for some $g \in \Gamma \setminus \{Id\}$.

It shows that the Möbius transformation $\gamma = g^{-1}$ maps s to another side of D(p)



Side-pairing transformations



In this figure, we always denote γ_s as the side pairing transformation associated to the side s.

And this transformation is $\gamma = g^{-1}$

Side-pairing transformations

Example(1)

Let $\Gamma = \{\gamma_n | \gamma_n(z) = z + n, n \in \mathbb{Z}\}$ be the Fuchsian group of integer translations

Let $\mathrm{p} = \mathrm{i}$, then $\mathrm{D(p)} = \left\{ z \in \mathbb{H} \middle| -\frac{1}{2} < Re(z) < \frac{1}{2} \right\}$ is a Dirichlet polygon for Γ.

Let s be the side and $\mathrm{s} = \left\{ z \in \mathbb{H} \middle| Re(z) = -\frac{1}{2} \right\}$

Let $\mathrm{g(z)} = \mathrm{z} - 1$, then s is perpendicular bisector of $[\mathrm{p}, \mathrm{p} - 1] = [\mathrm{p}, \mathrm{g(p)}]$

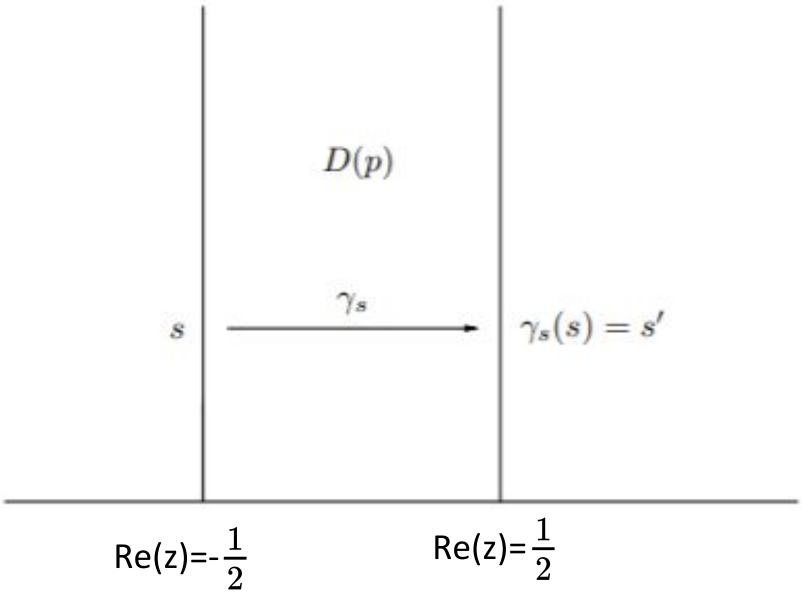
Since $\mathrm{g(p)} = \mathrm{p} - 1$

Then, $\gamma_s = g^{-1} = z + 1$

Therefore, γ_s is on the other side such that $\gamma_s = \left\{ z \in \mathbb{H} \middle| Re(z) = \frac{1}{2} \right\}$

Side-pairing transformations

Example(1) - continued



Side-pairing transformations

Recall

The modular group is defined to be

$$PSL(2,\mathbb{Z}) = \left\{ \frac{az+b}{cz+d} | a,b,c,d, \in \mathbb{Z}, ad-bc=1 \right\}$$

Let $\mathrm{k} > 1$ and let $\mathrm{p} = \mathrm{ki}$. The Dirichlet polygon for the modular group $\mathrm{PSL}(2,\mathbb{Z})$ is

$$D(p) = \left\{ z \in \mathbb{H} | |z| > 1, -\frac{1}{2} < Re(z) < \frac{1}{2} \right\}$$

Side-pairing transformations

Example(2)

Let $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ and we have $\mathrm{D(p)} = \left\{ \mathrm{z} \in \mathbb{H} | -\frac{1}{2} < \mathrm{Re(z)} < \frac{1}{2},\ |\mathrm{z}| > 1 \right\}$ and $\mathrm{p} = \mathrm{ik}$

The polygon have 3 sides

$$s_1 = \left\{ z \in \mathbb{H} | Re(z) = -\frac{1}{2}, |z| > 1 \right\}$$
$$s_2 = \left\{ z \in \mathbb{H} | Re(z) = \frac{1}{2}, |z| > 1 \right\}$$
$$s_3 = \left\{ z \in \mathbb{H} | -\frac{1}{2} < Re(z) < \frac{1}{2}, |z| = 1 \right\}$$

Side-pairing transformations

Example(2) - continued

By example , we know that $\gamma_{s_1} = z + 1$, and it pairs s_1 and s_2

On the other hand, the side pairing transformation associated to the side s_2 is $\gamma_{s_2} = z - 1$

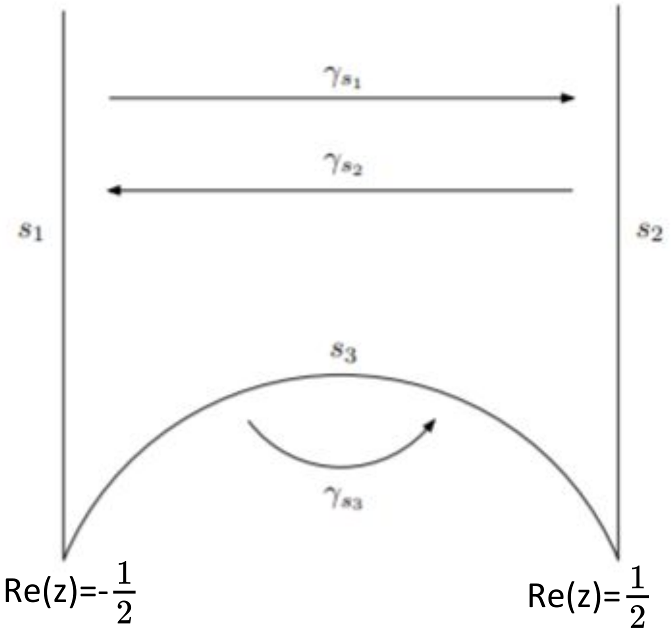
For s_3, it is perpendicular bisector of $[p, -1/p]$, the $\gamma_{s_3}^{-1}(p) = -1/p$,

Then, $\gamma_{s_3} = -1/z$

Note that, γ_{s_3} reverses the orientation of s_3

Side-pairing transformations

Example(2) - continued



Side-pairing transformations

Example(3)

Let $\Gamma = \{\gamma_n | \gamma_n(z) = 2^n z, n \in \mathbb{Z} \}$. Find the side pairing transformations for the Dirichlet polygon.

$$D(p) = \left\{ z \in \mathbb{H} \mid \frac{1}{\sqrt{2}} < Re(z) < \sqrt{2} \right\}$$

Side-pairing transformations

Example(3) Solution

Let $\mathrm{p = i}$ and let $\gamma_{\mathrm{n}}(\mathrm{z}) = 2^{\mathrm{n}}\mathrm{z}$.

There are two sides.

$$s_1 = \left\{ z \in \mathbb{C} | \, |z| = \frac{1}{\sqrt{2}} \right\}$$

$$s_2 = \left\{ z \in \mathbb{C} | \, |z| = \sqrt{2} \right\}$$

Since we have $\gamma_{-1}(\mathrm{p}) = 2^{-1}\mathrm{p} = \frac{1}{2}\mathrm{p}$, the s_1 is the perpendicular bisector of $[\mathrm{p}, \gamma_{-1}(\mathrm{p})]$

Hence,

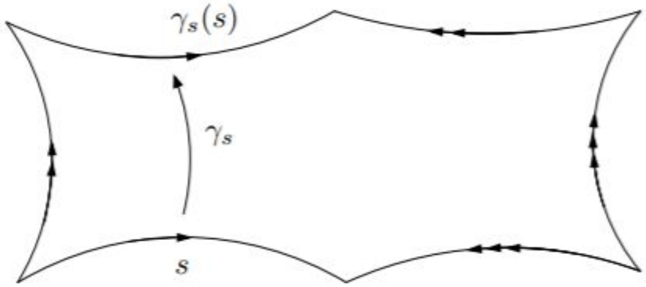
$$\gamma_{s_1}(z) = \gamma_{-1}^{-1}(z) = 2z$$

And,

$$\gamma_{s_2} = \gamma_{s_1}^{-1}(z) = \frac{z}{2}$$

Side-pairing transformations

Using a diagram to represent the side pairing transformation



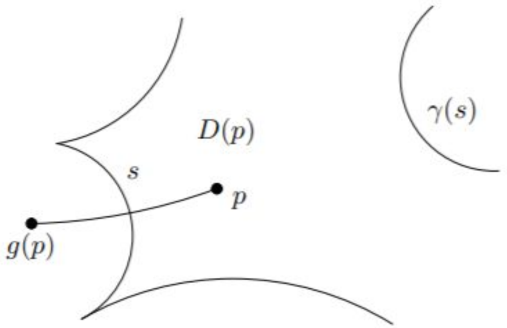
- The sides with an equal number of arrow are paired.
- The pairing preserves the direction of the arrows denoting the orientation of the sides.

Summary of side pairing transformation

- Way to find a side-pairing transformation associated to it

1. Construct D(p)
2. s is contained in the perpendicular bisector $L_p(g)$ of the geodesic segment [p, g(p)], for some $g \in \Gamma \setminus \{Id\}$.
3. Show $\gamma = g^{-1}$ maps s to another side of D(p)

- Use a diagram to represent side pairing transformation



2. Elliptic cycles

Recall: Side-pairing transformations

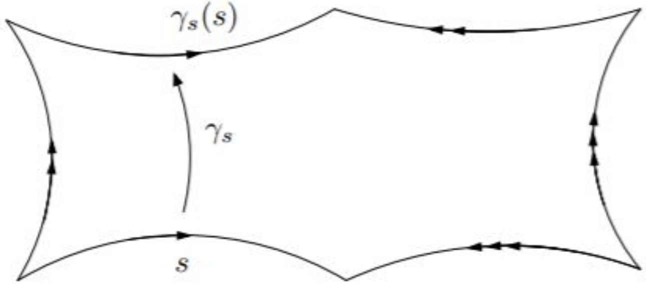
Definition

Let Γ be a Fuchsian group and let D(p) be a Dirichlet polygon for Γ.
We assume that D(p) has finitely many sides. Let s be a side of D.
Suppose that for some $\gamma \in \Gamma \setminus \{Id\}$
we have that γ(s) is also a side of D(p).
Note that $\gamma^{-1} \in \Gamma \setminus \{Id\}$ maps the side γ(s) back to the side s.

Then, the sides s and γ(s) are paired and call γ a **side pairing transformation**.

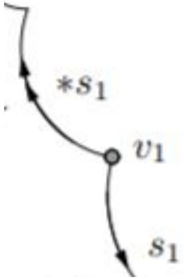
Recall: Side-pairing transformations

Using a diagram to represent the side pairing transformation



- The sides with an equal number of arrow are paired
- The pairing preserves the direction of the arrows denoting the orientation of the sides.

Elliptic cycles



Note that,

Each vertex v of D is mapped to another vertex of D under a side pairing transformation associated to a side with end point at v.

Each vertex v of D has two sides s and ∗s of D with end points at v. Let the pair (v, s) denote a vertex v of D and a side s of D with an endpoint at v. We denote by ∗(v, s) the pair comprising of the vertex v and the other side ∗s that ends at v.

Elliptic cycles

Definition

Let $v = v_0$ be a vertex of D and let s_0 be a side with an endpoint at v_0.

Let γ_1 be the side pairing transformation associated to the side s_0.

And it maps s_0 to another side s_1 of D

Let $s_1 = \gamma_1(s_0)$ and $v_1 = \gamma_1(v_0)$, and (v_1, s_1) is a new pair

Now, for $*(v_1,\ s_1)$,

Let γ_2 be the side pairing transformation associated to the side $*s_1$, and $\gamma_2(*s_1) = s_2$ and $\gamma_2(v_1) = v_2$

Note that v_2 is also a vertex of D

And repeat the above inductively

Elliptic cycles

Definition-continued

As there are only finitely many pairs (v, s),

this process of applying a side pairing transformation followed by applying $*$ must eventually return to the initial pair (v_0, s_0).

Let n be the least integer $n > 0$ for which $(v_n, *s_n) = (v_0, s_0)$.

$$\begin{pmatrix} v_0 \\ s_0 \end{pmatrix} \xrightarrow{\gamma_1} \begin{pmatrix} v_1 \\ s_1 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} v_1 \\ *s_1 \end{pmatrix}$$

$$\xrightarrow{\gamma_2} \begin{pmatrix} v_2 \\ s_2 \end{pmatrix} \xrightarrow{*} \cdots$$

$$\xrightarrow{\gamma_i} \begin{pmatrix} v_i \\ s_i \end{pmatrix} \xrightarrow{*} \begin{pmatrix} v_i \\ *s_i \end{pmatrix}$$

$$\xrightarrow{\gamma_{i+1}} \begin{pmatrix} v_{i+1} \\ s_{i+1} \end{pmatrix} \xrightarrow{*} \cdots.$$

Elliptic cycles

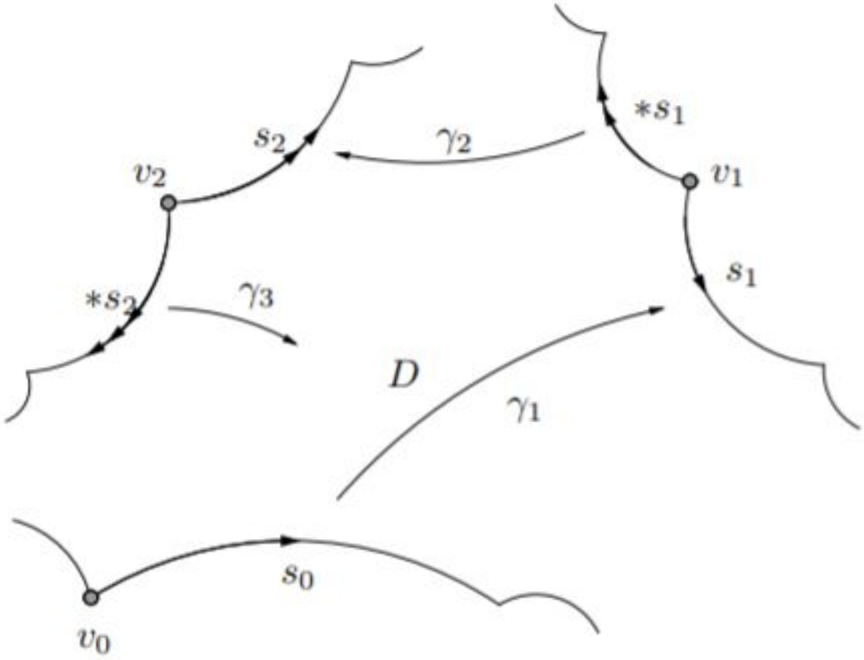
Definition-continued

The sequence of vertices $\mathcal{E} = v_0 \to v_1 \to \cdots \to v_{n-1}$ called an **elliptic cycle**.

The transformation $\gamma_n \gamma_{n-1} \ldots \gamma_2 \gamma_1$ is called an **elliptic cycle transformation**,

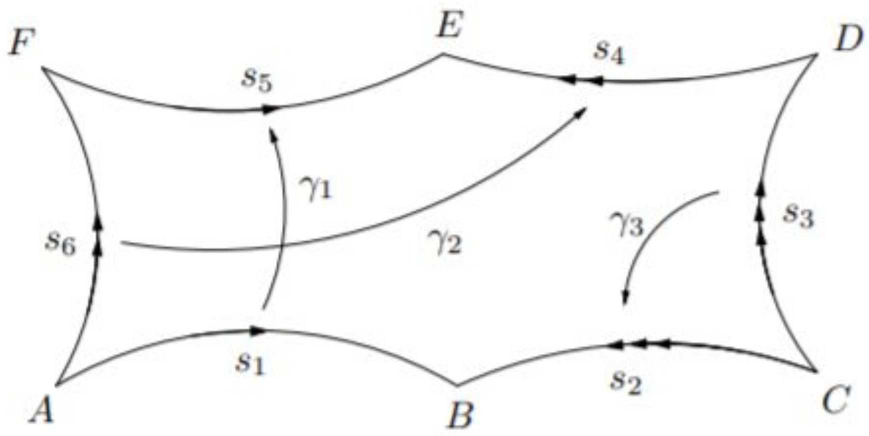
As there are only finitely many pairs of vertices and sides, we see that there are only finitely many elliptic cycles and elliptic cycle transformations.

Elliptic cycles



Elliptic cycles

Example



Elliptic cycles

Example-continued

There are two elliptic cycle in this figure.

The first one is A $\to$ F $\to$ E $\to$ B $\to$ D and the elliptic cycle transformation $\gamma_2^{-1}\gamma_3^{-1}\gamma_1^{-1}\gamma_2\gamma_1$

And here is the sequence of pairs of vertices and sides :

$$\begin{aligned}
\begin{pmatrix} A \\ s_1 \end{pmatrix} \overset{\gamma_1}{\to} & \begin{pmatrix} F \\ s_5 \end{pmatrix} \overset{*}{\to} \begin{pmatrix} F \\ s_6 \end{pmatrix} \\
\overset{\gamma_2}{\to} & \begin{pmatrix} E \\ s_4 \end{pmatrix} \overset{*}{\to} \begin{pmatrix} E \\ s_5 \end{pmatrix} \\
\overset{\gamma_1^{-1}}{\to} & \begin{pmatrix} B \\ s_1 \end{pmatrix} \overset{*}{\to} \begin{pmatrix} B \\ s_2 \end{pmatrix} \\
\overset{\gamma_3^{-1}}{\to} & \begin{pmatrix} D \\ s_3 \end{pmatrix} \overset{*}{\to} \begin{pmatrix} D \\ s_4 \end{pmatrix} \\
\overset{\gamma_2^{-1}}{\to} & \begin{pmatrix} A \\ s_6 \end{pmatrix} \overset{*}{\to} \begin{pmatrix} A \\ s_1 \end{pmatrix}.
\end{aligned}$$

Elliptic cycles

Example-continued

The another elliptic cycle is C with associated elliptic cycle transformation γ_3.

$$\begin{pmatrix} C \\ s_3 \end{pmatrix} \xrightarrow{\gamma_3} \begin{pmatrix} C \\ s_2 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} C \\ s_3 \end{pmatrix}$$

Elliptic cycle Transformation

Definition

Let v be a vertex of the hyperbolic polygon D.

We denote the **elliptic cycle transformation** associated to the vertex v and the side s by $\gamma_{v,s}$.

Elliptic cycles

Remark

1. Suppose we had started at (v, $*$s) instead of (v, s).

   Then we have an elliptic cycle transformation $\gamma_{v,*s}$.

   We also know that $\gamma_{v,\,s} = \gamma_{v,*s}^{-1}$

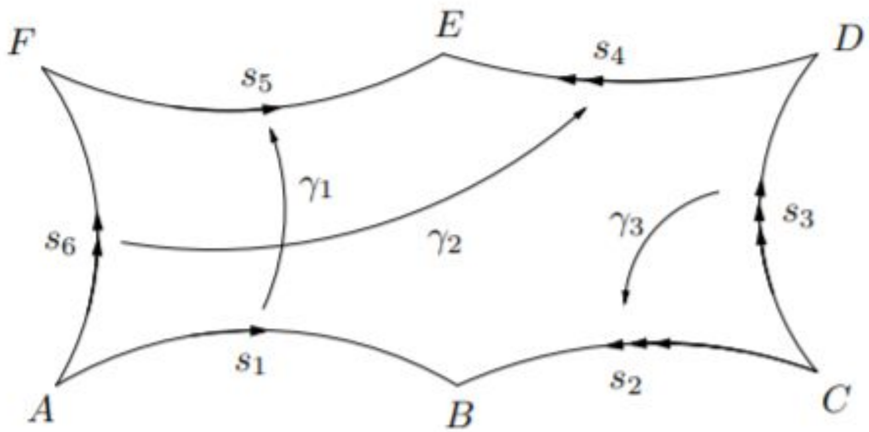
2. Suppose we started at (v_i, s_i) instead of (v_0, s_0)

   Then, the elliptic cycle transformation will become

   $$\gamma_{v_i,s_i} = \gamma_i \gamma_{i-1} \cdots \gamma_1 \gamma_n \cdots \gamma_{i+2} \gamma_{i+1}$$

Elliptic cycles

(Recall) Example



Order of an elliptic cycle

Definition

Let $\gamma \in$ Möbius transformation. We say that γ has finite order if there exists an integer m $>$ 0 such that $\gamma^m = \mathrm{Id}$

We call the smallest positive integer m to be the order of γ.

Order of an elliptic cycle

Proposition (1)

Let Γ be a Fuchsian group and let $\gamma \in \Gamma$ be an elliptic element. Then there exists an integer $k \geq 1$ such that $\gamma^k = \mathrm{Id}$

Proof of Proposition (1)

Recall a specific example of Fuchsian group in the upper half plane,

Let $\gamma(z) = \dfrac{\cos(\theta)z + \sin(\theta)}{-\sin(\theta)z + \cos(\theta)}$

be a rotation around i.

Order of an elliptic cycle

Proof of Proposition (1)-continued

Let $\gamma(z) = \dfrac{(\cos\theta)\ z + \sin\theta}{(-\sin\theta)z + \cos\theta}$

Then, $\gamma^m(z) = \dfrac{\cos(m\theta)\ z + \sin(m\theta)}{-\sin(m\theta)z + \cos(m\theta)}$, $m \geq 1$

Unless $\theta = \pi\ k$ for some $k \geq 1$, γ cannot be isolated and it is not in Fuchsian group

Then, $\theta = \pi\ k$ and then $\gamma^k = \mathrm{Id}$.

Order of an elliptic cycle

Proposition (2)

$\gamma_{v_0,s_0}, \gamma_{v_i,s_i}$ have the same power.

Proof of proposition

Suppose the elliptic vertex cycle is

$v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_{n-1}$

Then, the side pairing transformation is

$\gamma_{v_0,s_0} = \gamma_n \gamma_{n-1} \cdots \gamma_1$

Order of an elliptic cycle

Proof of Proposition - continued

Let the order of $\gamma_{\mathrm{v}_0,\mathrm{s}_0}$ is m and it is positive

For $\gamma_{\mathrm{v}_\mathrm{i},\mathrm{s}_\mathrm{i}}$, the elliptic cycle is

$$\gamma_{v_i,s_i} = \gamma_i\gamma_{i-1}\cdots\gamma_1\gamma_n\cdots\gamma_{i+1}$$

$$= (\gamma_i\gamma_{i-1}\cdots\gamma_1)\gamma_{v_0,s_0}(\gamma_i\cdots\gamma_1)^{-1}$$

Then,

$$\gamma_{v_i,s_i}^m = (\gamma_i\gamma_{i-1}\cdots\gamma_1)\gamma_{v_0,s_0}(\gamma_i\cdots\gamma_1)^{-1}(\gamma_i\gamma_{i-1}\cdots\gamma_1)\gamma_{v_0,s_0}(\gamma_i\cdots\gamma_1)^{-1}$$

$$\cdots(\gamma_i\gamma_{i-1}\cdots\gamma_1)\gamma_{v_0,s_0}(\gamma_i\cdots\gamma_1)^{-1}$$

$$= (\gamma_i\gamma_{i-1}\cdots\gamma_1)\gamma_{v_0,s_0}^m(\gamma_i\cdots\gamma_1)^{-1}$$

$$= (\gamma_i\gamma_{i-1}\cdots\gamma_1)(\gamma_i\cdots\gamma_1)^{-1}$$

$$= Id$$

Then, its order is m too.

Order of an elliptic cycle

Proposition (3)

If the order of γ is m, then the order of γ^{-1} is also m.

Proof of proposition

Suppose the order of γ is m.

Then,

$$\gamma^m = Id$$

$$\gamma ... \gamma \gamma^{-1} ... \gamma^{-1} = Id \cdot (\gamma^{-1})^m)$$

$$Id = (\gamma^{-1})^m$$

Then, the order of γ^{-1} is equal to m too.

Angle Sum

Definition

Let $\angle$ v be the internal angle of D at the vertex v.

The elliptic cycle ε: $v_0 \to v_1 \to \cdots \to v_{n-1}$ of the vertex $v = v_0$

We can write sum(ε) be the **angle sum**

$$sum(\varepsilon) = \angle v_0 + \ldots + \angle v_{n-1}$$

Angle Sum

Proposition

Let Γ be a Fuchsian group with Dirichlet polygon D with all vertices in $\mathbb{H}$

and let ε be an elliptic cycle.

Then, there exist some $m_\varepsilon \geq 1$ such that

$$m_\varepsilon sum(\varepsilon) = 2\pi$$

Accidental cycle.

Definition

If an elliptic cycle transformation is the identity then we call the elliptic cycle an accidental cycle.

Remark

The interior angle sum of an accidental elliptical cycle is 2π

Proof by the previous proposition, since it is an identity,

then, $m_\varepsilon = 1$ and $\text{sum}(\varepsilon) = 2\pi$

Summary of Elliptic cycles

1. Elliptic cycle
2. Elliptic cycle transformation
3. Order of elliptic cycle
4. Angle sum
5. Accidental cycle

3. Generators and Relations

Recall

Definition

<u>Additive Group</u>

Binary Operation:	addition (+)
Identity element:	0
Inverse of the element a:	-a

<u>Multiplicative Group</u>

Binary Operation:	multiplication (·)
Identity element:	1
Inverse of the element g:	g^{-1}

<u>Group</u>

1. Closure
2. Associativity
3. Identity
4. Inverse

Generater

Definition

Let Γ be a group.

We say that a subset $S = \{\gamma_1, \ldots, \gamma_n\} \subset \Gamma$ is **a set of generators** if every element of Γ can be written as a composition of elements from S and their inverses.

We write $\Gamma = \langle S \rangle$.

Generater

Example(1)

1 is a generator of the additive group $\mathbb{Z}$.

Let n $\in$ $\mathbb{Z}$,

Case 1: $n > 0$ can be written as $1+\cdots+1$ (n times)

Case 2: $n < 0$ can be written as $(-1)+\cdots+(-1)$ (-n times)

Case 3: $n = 0$ can be written as $(-1) + 1$

Generater

Example(2)

{(1, 0), (0, 1)} is a set of generators of the additive group $\mathbb{Z}^2 = \{(n, m) \mid n, m \in \mathbb{Z}\}$.

Example(3)

$\omega = e^{2\pi i/p}$ is a generator of the multiplicative group of pth roots of unity $\{1, \omega, \ldots, \omega^{p-1}\}$.

Generater

Remark

In general, a group have many different generating sets.

e.g. {2,3} , {314,315} are sets of generators of $\mathbb{Z}$.

Note that 1 = 3 - 2,

Hence n = 3 + · · · + 3 + (-2) + · · · + (-2) where there are n 3s and n -2s.

Generater

Theroem

Let Γ be a Fuchsian group.

Suppose that D(p) is a Dirichlet polygon with $\mathrm{Area}_{\mathbb{H}}(D(p)) < \infty$.

Then the set of side-pairing transformations of D(p) generate Γ.

Generater

Example

Let Γ be $\mathrm{PSL}(2,\mathbb{Z}) = \left\{ \gamma(z) = \frac{az+b}{cz+d} \mid a,b,c,d \in \mathbb{Z},\ ad-bc=1 \right\}$.

A fundamental domain for Γ is $D(p) = \{z \in \mathbb{H} \mid |z| > 1,\ -1/2 < \mathrm{Re}(z) < 1/2\}$.
where $p = ik$ for any $k > 1$.

Recall that the side-pairing transformations are $z \mapsto z+1$ and $z \mapsto -1/z$.

Then followed by the theorem,

$$\mathrm{PSL}(2,\mathbb{Z}) = \langle z \mapsto z+1, z \mapsto -1/z \rangle$$

Word

Definition

Let S be a finite set of k symbols.

Let $S^{-1} = \{ a^{-1} \mid a \in S \}$.

Consider the concatenation of symbols chosen from $S \cup S^{-1}$,

subject to the condition that concatenations of the form aa^{-1} and $a^{-1}a$ are removed.

Such a finite concatenation of n symbols is called a **word** of length n.

Word

Example

Let S = { a , h , n , o }.

Then $S \cup S^{-1}$ = { a , h , n , o , a^{-1} , h^{-1} , n^{-1} , o^{-1} }

The followings are word in S:

ah, no, $n^{-1}o^{-1}$, nanoha, $o^{-1}o^{-1}o^{-1}$hhhh, e(empty word)

The followings are not word in S:

oo^{-1} , $aaah^{-1}$hhh, cuhk

Free Group

Definition

Let $$\begin{aligned} \mathcal{W}_n &= \{\text{all words of length } n\} \\ &= \{w_n = a_1 \cdots a_n \mid a_j \in S \cup S^{-1},\ a_{j\pm1} \neq a_j^{-1}\}. \end{aligned}$$

Let e denote the empty word and $\mathcal{W}_0 = \{e\}$.

We define $\mathcal{F}_k = \bigcup_{n \geq 0} \mathcal{W}_n$ to be the **free group** on k generators.

Free Group

The free group is a group.

Proof

1) Well-defined: The concatenation of two words is another word.

2) Associative: The concatenation is associative by observation.

3) Existence of an identity: The empty word e is the identity element such that if $w = a_1 \cdots a_n \in \mathcal{F}_n$ then $we = ew = w$.

4) Existence of inverses: If $w = a_1 \cdots a_n$ is a word, then $w^{-1} = a_n^{-1} \cdots a_1^{-1}$ such that $ww^{-1} = w^{-1}w = e$.

Generator and Relation

Definition

Let $S = \{a_1, \ldots, a_k\}$ be a finite set of symbols, $w_1, \ldots, w_m$ be a finite set of words,

We define the group $\Gamma = \langle a_1, \ldots, a_k \mid w_1 = \ldots = w_m = e \rangle$

to be the set of all words of symbols from $S \cup S^{-1}$,

subject to the following conditions:

1) any subwords of the form aa^{-1} or $a^{-1}a$ are deleted
2) any occurrences of the subwords $w_1, \ldots, w_m$ are deleted.

We call the above group Γ the group with **generators** $a_1, \ldots, a_k$ and **relations** $w_1, \ldots, w_m$.

Isomorphism

Definition

Let Γ_1, Γ_2 be two groups.

A map $\phi : \Gamma_1 \to \Gamma_2$ is an isomorphism if

1) ϕ is a bijection (surjective + injective)
2) $\phi(\gamma_1\gamma_2) = \phi(\gamma_1)\phi(\gamma_2) \; \forall \gamma_1, \gamma_2 \in \Gamma_1$

We say that Γ_1, Γ_2 are isomorphic.

Finitely Presented

Definition

We say that a group Γ is finitely presented if it is isomorphic to a group in the form:

$$\langle a_1, \ldots, a_k \mid w_1 = \ldots = w_m = e \rangle$$

with finitely many generators and finitely many relations.

We say $\langle a_1, \ldots, a_k \mid w_1 = \ldots = w_m = e \rangle$ is a presentation of Γ.

Finitely Presented

Example (1)

The free group on k generators $\mathcal{F}_k = \bigcup_{n \geq 0} \mathcal{W}_n$ is finitely presented,

where
$$\begin{aligned} \mathcal{W}_n &= \{\text{all words of length } n\} \\ &= \{w_n = a_1 \cdots a_n \mid a_j \in S \cup S^{-1},\ a_{j\pm 1} \neq a_j^{-1}\}. \end{aligned}$$

There are no relations for the free group on k generators.

Finitely Presented

Example(2)

The multiplicative group of pth roots of unity $\{1, \omega, \ldots, \omega^{p-1}\}$ is finitely presented

where $\omega = e^{2\pi i/p}$.

Using the group isomorphism $\omega \mapsto a$, we can write it in the form:

$$\langle a \mid a^p = e \rangle.$$

Finitely Presented

Example(3)

The additive group $\mathbb{Z}$ is finitely presented.

It is actually the free group on one generator: $\langle a \rangle = \{a^n \mid n \in \mathbb{Z}\}.$

Note that $a^{n+m} = a^n a^m \quad \forall\, a \in \mathbb{Z}$,

Hence $\langle a \rangle$ is isomorphic to $\mathbb{Z}$ under the isomorphism $a^n \mapsto n$.

Finitely Presented

Example(4)

The additive and abelian group $\mathbb{Z}^2 = \{(n,m) \mid n,m \in \mathbb{Z}\}$ is finitely presented.

The free group ⟨a, b⟩ is not abelian because ab ≠ ba.

$$\begin{aligned} ba = bae &= ba(a^{-1}b^{-1}ab) \\ &= b(aa^{-1})b^{-1}ab \\ &= beb^{-1}ab \\ &= bb^{-1}ab \\ &= ab. \end{aligned}$$

Hence, we add the relation $a^{-1}b^{-1}ab$ such that $\langle a, b \mid a^{-1}b^{-1}ab = e\rangle = \{a^n b^m \mid n,m \in \mathbb{Z}\}$

Using the group isomorphism $(n,m) \mapsto a^n b^m$, the group is isomorphic to $\mathbb{Z}^2$

Finitely Presented

Example(5)

The group $\langle a, b \mid a^4 = b^2 = (ab)^2 = e \rangle$ is finitely presented.

By computation, the elements in this group are: $e, a, a^2, a^3, b, ab, a^2b, a^3b.$

This is actually the dihedral group.

a : an anti-clockwise rotation through a right-angle

b : reflection in a diagonal

Summary

1) Generator
2) Word
3) Free Group
4) Isomorphism
5) Finitely Presented

4. Poincaré’s Theorem

Recall

Elliptic cycle

The sequence of vertices $\mathcal{E} = v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_{n-1}$ is called an elliptic cycle.

Elliptic cycle transformation

The transformation $\gamma_n \gamma_{n-1} \cdots \gamma_2 \gamma_1$ is called an elliptic cycle transformation.

Let v be a vertex of the hyperbolic polygon D and let s be a side of D with an end-point at v.

We denote the elliptic cycle transformation associated to the pair (v,s) by $\gamma_{v,s}$.

Recall

Angle Sum

Let $\angle$ v be the internal angle of D at the vertex v.

The elliptic cycle ε $v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_{n-1}$ of the vertex $v = v_0$

We can write sum(ε) be the **angle sum**

$$sum(\varepsilon) = \angle v_0 + \ldots + \angle v_{n-1}$$

Recall

Elliptic Cycle condition

An elliptic cycle $\mathcal{E}$ satisfies the elliptic cycle condition if there exists an integer m ≥ 1, depending on $\mathcal{E}$ such that

$$m\,\mathrm{sum}(\mathcal{E}) = 2\pi.$$

Recall

Half-plane

Let C be a geodesic in $\mathbb{H}$. Then C divides $\mathbb{H}$ into two components. These components are called **half-planes**.

Convex Hyperbolic Polygon

A **convex hyperbolic polygon** is the intersection of a finite number of half-planes.

Poincaré's Theorem (no boundary vertices)

Let D be a convex hyperbolic polygon with finitely many sides.

Suppose:

1) All vertices lie inside $\mathbb{H}$ and that D is equipped with a collection $\mathcal{G}$ of side-pairing Möbius transformations.

2) No side of D is paired with itself.

3) The elliptic cycles are $\mathcal{E}_1, \ldots, \mathcal{E}_r$.

4) Each elliptic cycle $\mathcal{E}_j$ of D satisfies the elliptic cycle condition: for each $\mathcal{E}_j$ there exists an integer $m_j \geq 1$ such that $m_j \, \mathrm{sum}(\mathcal{E}_j) = 2\pi$.

Poincaré’s Theorem (no boundary vertices)

Then:

1) The subgroup $\Gamma = \langle \mathcal{G} \rangle$ generated by $\mathcal{G}$ is a Fuchsian group.

2) The Fuchsian group Γ has D as a fundamental domain.

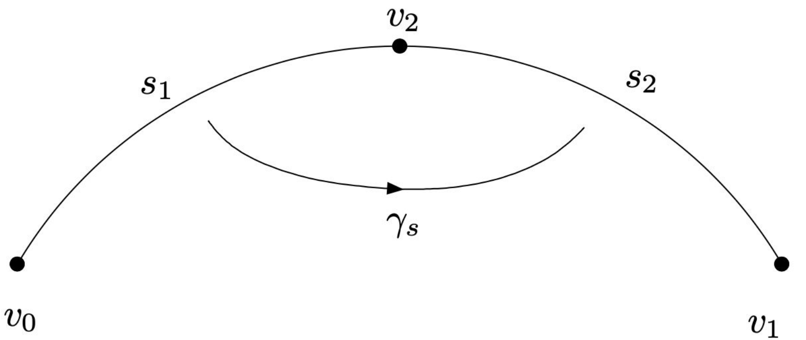
3) The Fuchsian group Γ can be written in terms of generators and relations as follows.
Think of $\mathcal{G}$ as an abstract set of symbols.
For each elliptic cycle $\mathcal{E}_j$, choose a corresponding elliptic cycle transformation
$\gamma_j = \gamma_{v,s}$ (for some vertex v on the elliptic cycle)
This is a word in symbols chosen from $\mathcal{G} \cup \mathcal{G}^{-1}$.
Then Γ is isomorphic to the group with generators $\gamma_s \in \mathcal{G}$ and relations cycle $\gamma_j^{m_j}$:

$$\Gamma = \langle \gamma_s \in \mathcal{G} \mid \gamma_1^{m_1} = \gamma_2^{m_2} = \cdots = \gamma_r^{m_r} = e \rangle.$$

Poincaré's Theorem (no boundary vertices)

Remark:
The second hypothesis that "No side of D is paired with itself" is not a real restriction.

We can introduce another vertex on the mid-point of that self-paired side,
thus dividing the side into two smaller sides which are then paired with each other.



The side s is paired with itself. By splitting it in half, we have two distinct sides that are paired.

Recall

Corollary

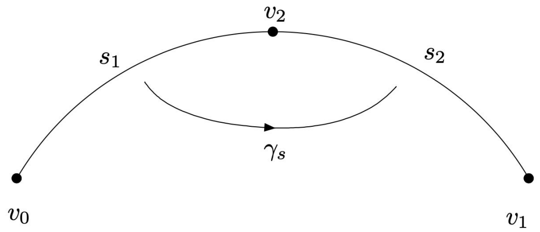
Suppose γ is a Möbius transformation of $\mathbb{H}$ with three or more fixed points.

Then γ is the identity (and so fixes every point).

Poincaré's Theorem (no boundary vertices)

Remark example:

Suppose that s is a side with side-pairing transformation γ_s that pairs s with itself.
Suppose that s has end-points at the vertices v_0 and v_1.
Introduce a new vertex v_2 at the mid-point of $[v_0, v_1]$.
Notice that $\gamma_s(v_2) = v_2$.
We must have that $\gamma_s(v_0) = v_1$ and $\gamma_s(v_1) = v_0$ (by the Corollary).
Let s_1 be the side $[v_0, v_2]$ and let s_2 be the side $[v_2, v_1]$.
Then $\gamma_s(s_1) = s_2$ and $\gamma_s(s_2) = s_1$.
Hence γ_s pairs the sides s_1 and s_2.
Notice that the internal angle at the vertex v_2 is equal to π.



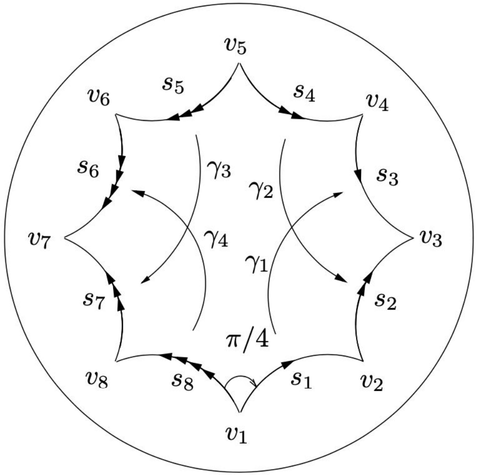
Poincaré's Theorem (no boundary vertices)

Example:

Consider a regular hyperbolic octagon with each internal angle equal to π/4 in $\mathbb{D}$.

Label the vertices of such an octagon anti-clockwise $v_1, \ldots, v_8$.

Label the sides anti- clockwise $s_1, \ldots, s_8$ so that side s_j occurs immediately after vertex v_j .



Poincaré's Theorem (no boundary vertices)

Example - continue

$$\begin{pmatrix} v_1 \\ s_1 \end{pmatrix} \overset{\gamma_1}{\to} \begin{pmatrix} v_4 \\ s_3 \end{pmatrix} \overset{*}{\to} \begin{pmatrix} v_4 \\ s_4 \end{pmatrix}$$

$$\overset{\gamma_2}{\to} \begin{pmatrix} v_3 \\ s_2 \end{pmatrix} \overset{*}{\to} \begin{pmatrix} v_3 \\ s_3 \end{pmatrix}$$

$$\overset{\gamma_1^{-1}}{\to} \begin{pmatrix} v_2 \\ s_1 \end{pmatrix} \overset{*}{\to} \begin{pmatrix} v_2 \\ s_2 \end{pmatrix}$$

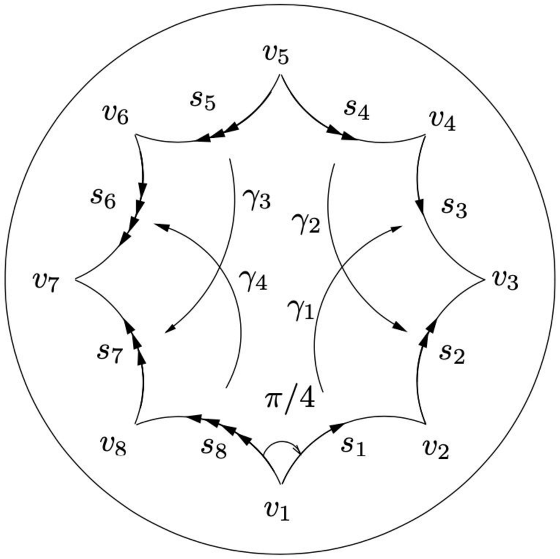
$$\overset{\gamma_2^{-1}}{\to} \begin{pmatrix} v_5 \\ s_4 \end{pmatrix} \overset{*}{\to} \begin{pmatrix} v_5 \\ s_5 \end{pmatrix}$$

$$\overset{\gamma_3}{\to} \begin{pmatrix} v_8 \\ s_7 \end{pmatrix} \overset{*}{\to} \begin{pmatrix} v_8 \\ s_8 \end{pmatrix}$$

$$\overset{\gamma_4}{\to} \begin{pmatrix} v_7 \\ s_6 \end{pmatrix} \overset{*}{\to} \begin{pmatrix} v_7 \\ s_7 \end{pmatrix}$$

$$\overset{\gamma_3^{-1}}{\to} \begin{pmatrix} v_6 \\ s_5 \end{pmatrix} \overset{*}{\to} \begin{pmatrix} v_6 \\ s_6 \end{pmatrix}$$

$$\overset{\gamma_4^{-1}}{\to} \begin{pmatrix} v_1 \\ s_8 \end{pmatrix} \overset{*}{\to} \begin{pmatrix} v_1 \\ s_1 \end{pmatrix}$$



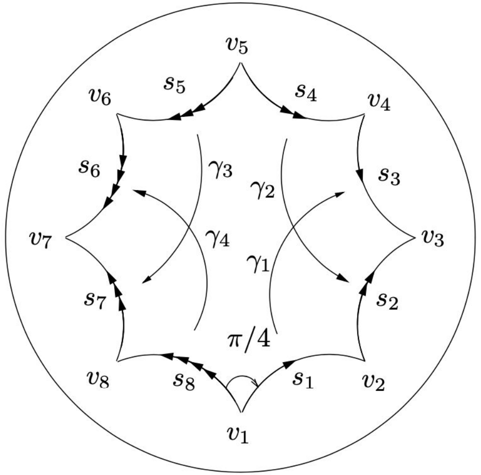
Therefore, there is just one elliptic cycle: $\mathcal{E} = v_1 \to v_4 \to v_3 \to v_2 \to v_5 \to v_8 \to v_7 \to v_6.$

with associated elliptic cycle transformation: $\gamma_4^{-1}\gamma_3^{-1}\gamma_4\gamma_3\gamma_2^{-1}\gamma_1^{-1}\gamma_2\gamma_1$

Poincaré's Theorem (no boundary vertices)

Example - continue

The internal angle at each vertex is π/4,
Then, the angle sum is 8π/4 =2π.
Hence the elliptic cycle condition holds (with $m_{\mathcal{E}}$ = 1).



By Poincaré's Theorem,
the group generated by $\gamma_1, \ldots, \gamma_4$ generate a Fuchsian group.

We can write this group in terms of generators and relations as follows:

$$\langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \mid \gamma_4^{-1}\gamma_3^{-1}\gamma_4\gamma_3\gamma_2^{-1}\gamma_1^{-1}\gamma_2\gamma_1 = e \rangle.$$

Recall

Half-plane

Let C be a geodesic in $\mathbb{H}$. Then C divides $\mathbb{H}$ into two components. These components are called **half-planes**.

Convex Hyperbolic Polygon

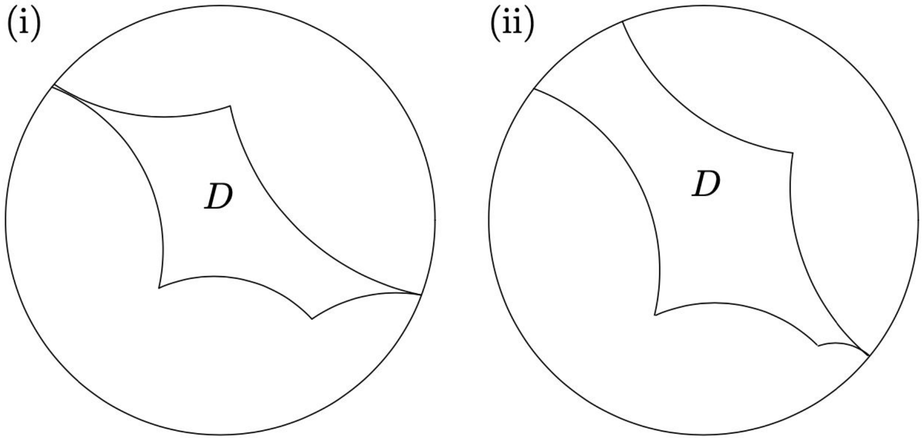
A **convex hyperbolic polygon** is the intersection of a finite number of half-planes.

Free Edge

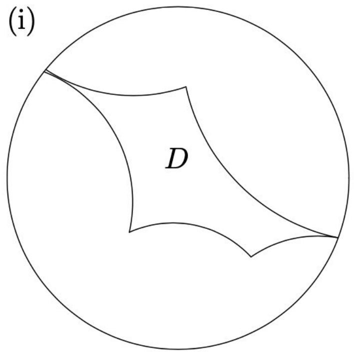
Definition

When a convex hyperbolic polygon has an edge lying on the boundary,

such edge is call a **free edge**.



Parabolic Cycle Transformation



Let D be a convex hyperbolic polygon with no free edges.

Suppose that each side s of D is equipped with a side-pairing transformation γ_s.

Suppose the half-plane bounded by s containing D is mapped by the isometry γ_s to the half-plane bounded by $\gamma_s(s)$ but opposite D.

Möbius transformations of $\mathbb{H}$ act on $\partial\mathbb{H}$ and map $\partial\mathbb{H}$ to itself.

Each side-pairing transformation maps a boundary vertex to another boundary vertex.

Parabolic Cycle Transformation

Definition

Let $v = v_0$ be a boundary vertex of D and let $s = s_0$ be a side with an end-point at v .

We call $\mathcal{P} = v_0 \to \cdots \to v_{n-1}$ a **parabolic cycle**

with associated **parabolic cycle transformation** $\gamma_{v,s} = \gamma_n \cdots \gamma_1$.

Parabolic Cycle Transformation

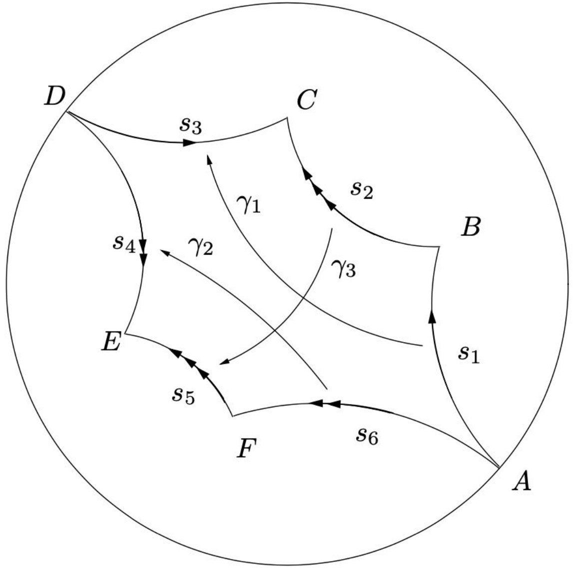
Example

$$\begin{pmatrix} A \\ s_1 \end{pmatrix} \xrightarrow{\gamma_1} \begin{pmatrix} D \\ s_3 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} D \\ s_4 \end{pmatrix}$$
$$\xrightarrow{\gamma_2^{-1}} \begin{pmatrix} A \\ s_6 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} A \\ s_1 \end{pmatrix}$$

Hence we have a parabolic cycle $A \rightarrow D$
with associated parabolic cycle transformation $\gamma_2^{-1}\gamma_1$.

$$\begin{pmatrix} B \\ s_2 \end{pmatrix} \xrightarrow{\gamma_3} \begin{pmatrix} F \\ s_5 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} F \\ s_6 \end{pmatrix}$$
$$\xrightarrow{\gamma_2} \begin{pmatrix} E \\ s_4 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} E \\ s_5 \end{pmatrix}$$
$$\xrightarrow{\gamma_3^{-1}} \begin{pmatrix} C \\ s_2 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} C \\ s_3 \end{pmatrix}$$
$$\xrightarrow{\gamma_1^{-1}} \begin{pmatrix} B \\ s_1 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} B \\ s_2 \end{pmatrix}$$

Hence we have the elliptic cycle $B \rightarrow F \rightarrow E \rightarrow C$
with associated elliptic cycle transformation $\gamma_1^{-1}\gamma_3^{-1}\gamma_2\gamma_3$.



Parabolic Cycle Condition

Definition

A parabolic cycle $\mathcal{P}$ satisfies the **<u>parabolic cycle condition</u>** if for some (hence all) vertex $v \in \mathcal{P}$, the parabolic cycle transformatior $\gamma_{v,s}$ is either a parabolic Möbius transformation or the identity.

Recall

Let γ be a Möbius transformation of $\mathbb{H}$.

If γ has:	Then γ is:
Three or more fixed points	The identity
Two fixed points in $\partial\mathbb{H}$ and none in $\mathbb{H}$	Hyperbolic
One fixed point in $\partial\mathbb{H}$ and none in $\mathbb{H}$	Parabolic
One fixed point in $\mathbb{H}$ and none in $\partial\mathbb{H}$	Elliptic

Poincaré's Theorem (with boundary vertices)

Let D be a convex hyperbolic polygon with finitely many sides,
possibly with boundary vertices but without free edges.

Suppose:

1) All vertices lie inside $\mathbb{H}$ and that D is equipped with a collection $\mathcal{G}$ of side-pairing Möbius transformations.

2) No side of D is paired with itself.

3) The elliptic cycles are $\mathcal{E}_1, \ldots, \mathcal{E}_r$ and the parabolic cycles are $\mathcal{P}_1, \ldots, \mathcal{P}_s$.

4) Each elliptic cycle $\mathcal{E}_j$ satisfies the elliptic cycle condition

5) Each parabolic cycle $\mathcal{P}_j$ satisfies the parabolic cycle condition.

Poincaré’s Theorem (with boundary vertices)

Then:

1) The subgroup $\Gamma = \langle \mathcal{G} \rangle$ generated by $\mathcal{G}$ is a Fuchsian group.

2) The Fuchsian group Γ has D as a fundamental domain.

3) The Fuchsian group Γ can be written in terms of generators and relations as follows.
Think of $\mathcal{G}$ as an abstract set of symbols.
For each elliptic cycle $\mathcal{E}_j$, choose a corresponding elliptic cycle transformation
$\gamma_j = \gamma_{v,s}$ (for some vertex v on the elliptic cycle)
This is a word in symbols chosen from $\mathcal{G} \cup \mathcal{G}^{-1}$.
Then Γ is isomorphic to the group with generators $\gamma_s \in \mathcal{G}$ and relations cycle $\gamma_j^{m_j}$:

$$\Gamma = \langle \gamma_s \in \mathcal{G} \mid \gamma_1^{m_1} = \gamma_2^{m_2} = \cdots = \gamma_r^{m_r} = e \rangle.$$

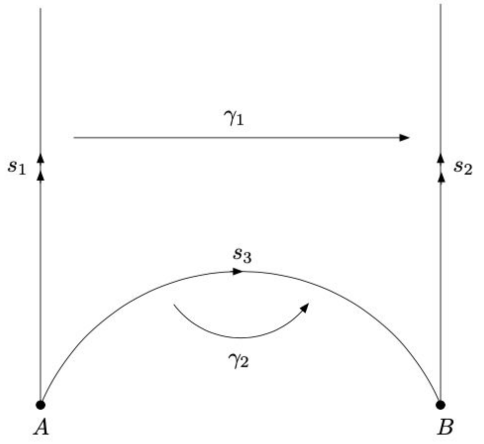
Poincaré's Theorem (with boundary vertices)

Example: $\mathrm{PSL}(2, \mathbb{Z})$

Let $A = (-1 + i\sqrt{3})/2$ and $B = (1 + i\sqrt{3})/2$.

The side pairing transformations are given by:
$\gamma_1(z) = z + 1$ and $\gamma_2(z) = -1/z$.

Notice that $\gamma_2(A) = B$ and $\gamma_2(B) = A$.



Side pairing transformations for the modular group

Poincaré’s Theorem (with boundary vertices)

Example - continue

The side $[A, B]$ is paired with itself by γ_2.
We introduce an extra vertex $C = i$ at the mid-point of $[A, B]$
B]

$$\begin{pmatrix} C \\ s_3 \end{pmatrix} \xrightarrow{\gamma_2} \begin{pmatrix} C \\ s_4 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} C \\ s_3 \end{pmatrix}$$

$$\mathcal{E}_2 = C$$

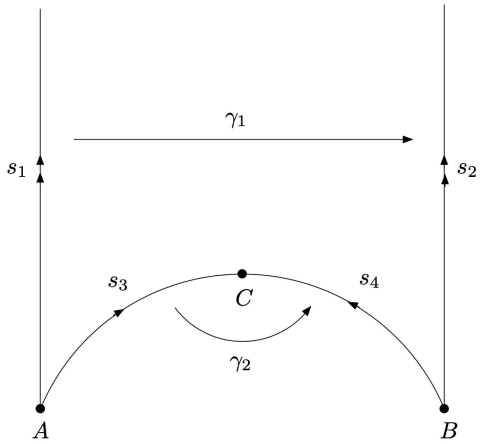
Hence we have an elliptic cy γ_2: E1 which has
elliptic cycle transformation .

$$2\angle C = 2\pi.$$

The angle sum of this elliptic cycle satisfies

$$m_2$$

Hence the elliptic cycle condition holds with m1 = 2.



Poincaré's Theorem (with boundary vertices)

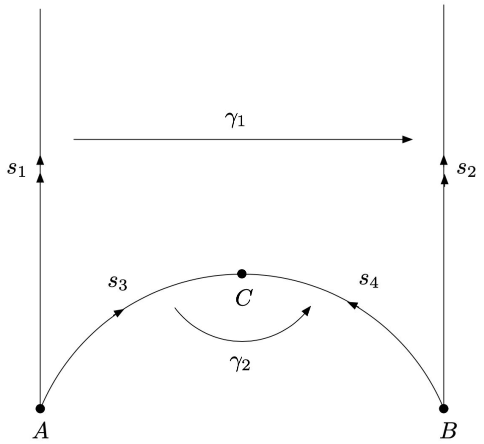
Example - continue

$$\begin{pmatrix} A \\ s_1 \end{pmatrix} \xrightarrow{\gamma_1} \begin{pmatrix} B \\ s_2 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} B \\ s_4 \end{pmatrix}$$
$$\xrightarrow{\gamma_2^{-1}} \begin{pmatrix} A \\ s_3 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} A \\ s_1 \end{pmatrix}$$

Hence $A \to B$ is an elliptic cycle $\mathcal{E}_1$ which has elliptic cycle transformation $\gamma_2^{-1}\gamma_1(z) = (-z-1)/z.$

The angle sum of this elliptic cycle satisfies
$$3(\angle A + \angle B) = 3(\pi/3 + \pi/3) = 2\pi.$$

Hence the elliptic cycle condition holds with $m_1 = 3$.



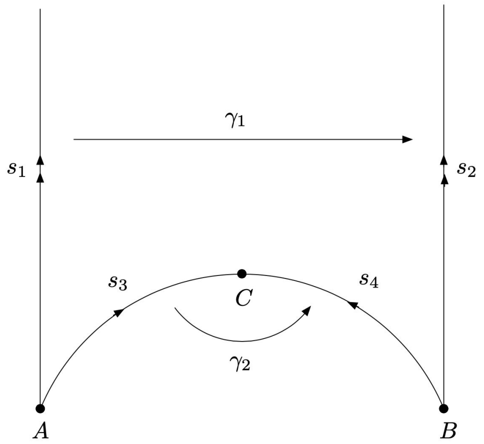
Poincaré's Theorem (with boundary vertices)

Example - continue

Hence we have a parabolic cycle ∞
with parabolic cycle transformation $\gamma_1(z) = z + 1$.
As γ_1 has a single fixed point at ∞, it is parabolic.
Hence the parabolic cycle condition holds.

By Poincaré's Theorem,
the group generated by γ_1 and γ_2 is a Fuchsian group.
Let $a = \gamma_1, \; b = \gamma_2.$
Then the group generated by γ_1, γ_2 in terms of generators and relations is as follows:

$$\mathrm{PSL}(2, \mathbb{Z}) = \langle a, b \mid (b^{-1}a)^3 = b^2 = e \rangle.$$



Summary

1) Poincaré’s Theorem (no boundary vertices)
2) Free Edge
3) Parabolic Cycle
4) Poincaré’s Theorem (with boundary vertices)

5. Reference

Reference

1. Lecture notes by C. Walkden
2. Hyperbolic geometry, by James W. Anderson, Springer, 1999.
3. Notes by Pollicott

Thank you very much!