Prop 3.5 Let $\Omega \subset \mathbb{C}$ be a region, &

- $\{f_n\}$, f be holo functions on Ω such that
- $f_n \to f$ uniformly on every compact subset of Ω

If f_n are injective, then

f is either injective or constant.

Pf: Suppose that f is not injective.

Then $\exists\ z_1, z_2 \in \Omega$ such that

$$z_1 \neq z_2 \text{ but } f(z_1) = f(z_2).$$

Define $g_n(z) = f_n(z) - f_n(z_1)$.

Then $\begin{cases} g_n(z_1) = 0 \ \& \\ g_n(z) \neq 0, \quad \forall z \in \Omega \setminus \{z_1\}. \end{cases}$

since f_n injectives

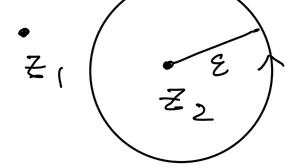
As $f_n \to f$ uniformly on cpt subset,

$g_n \to g = f - f(z_1)$ uniformly on cpt. subset.

If $g \not\equiv 0$, then z_2 is an isolated zero of g.

$(g(z_2) = f(z_2) - f(z_1) = 0)$

$$\Rightarrow \quad 1 \leq \frac{1}{2\pi i} \int_{|\zeta - z_2| = \varepsilon} \frac{g'(\zeta)}{g(\zeta)}\, d\zeta$$



along a small circle $|\zeta - z_2| = \varepsilon$ around z_2

s.t. $g(\zeta) \neq 0$, $\forall\ |\zeta - z_2| \leq \varepsilon$.

Then $\frac{1}{g_n} \to \frac{1}{g}$ uniformly on $|\zeta - z_2| = \varepsilon$

and hence $\frac{1}{2\pi i}\int_{|\zeta-z_2|=\varepsilon}\frac{g_n'(\zeta)}{g_n(\zeta)}d\zeta \to \frac{1}{2\pi i}\int_{|\zeta-z_2|=\varepsilon}\frac{g'(\zeta)}{g(\zeta)}d\zeta \geq 1$

This is a contradiction as g_n has no zero in $|\zeta-z_2|\leq\varepsilon$

$$\Rightarrow \quad \frac{1}{2\pi i}\int_{|\zeta-z_2|=\varepsilon}\frac{g_n'(\zeta)}{g_n(\zeta)}d\zeta = 0 \quad , \quad \forall n,$$

$\therefore \quad g\equiv 0 \quad \Rightarrow \quad f(z)=f(z_1)$ a constant $\forall z\in\Omega$ ※

<u>Remark</u>: The argument in the proof of Prop 3.5 gives the following

<u>Hurwitz Theorem</u>:
If f_n & f analytic in Ω, $f_n(z)\neq 0$, $\forall z\in\Omega$, and f_n converges uniformly to f on every compact set of Ω, then either (i) $f(z)\equiv 0$, or (ii) $f(z)\neq 0$, $\forall z\in\Omega$.

And clearly Hurwitz Thm $\Rightarrow$ Prop 3.5.

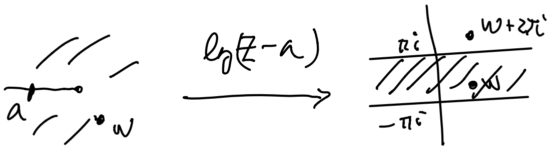
3.3 <u>Proof of the Riemann Mapping Theorem</u>

<u>Step 1</u> For a <u>proper</u> and <u>simply-connected</u> <u>region</u> Ω, and $z_0\in\Omega$, $\exists$ <u>conformal</u>

$f:\Omega\to \underline{f(\Omega)\subset D}$ s.t. $\underline{f(z_0)=0}$ & $\underline{f'(z_0)>0}$

Pf: Ω is proper $\Rightarrow \exists\, a \in \mathbb{C} \setminus \Omega$.

Then Ω is simply-connected $\Rightarrow$



$g(z) = \log(z-a)$ is well-defined in Ω.

Clearly (i) g is injective

(ii) $\forall\, w \in \Omega$, $g(z) \neq g(w) + 2\pi i$, $\forall\, z \in \Omega$

(by taking exponential)

Claim: $\forall\, w \in \Omega$, $\exists\, r > 0$ s.t. $\overline{D}_r(g(w)+2\pi i) \cap g(\Omega) = \emptyset$.

Pf: Suppose not, then $\exists\, z_n \in \Omega$ such that

$$g(z_n) \to g(w) + 2\pi i.$$

Taking exponential, $z_n \to w$.

And hence $g(z_n) \to g(w)$

which is a contradiction.

Then $$h(z) = \frac{r}{g(z) - (g(w)+2\pi i)}$$ is holo. injective

and $$|h(z)| = \frac{r}{|g(z) - (g(w)+2\pi i)|} < \frac{r}{r} = 1$$

$\therefore\ h : \Omega \to h(\Omega) \subset \mathbb{D}$ conformal

Finally, $$f(z) = e^{i\theta} \frac{h(z_0) - h(z)}{1 - \overline{h(z_0)}\, h(z)} = e^{i\theta}\, \psi_{h(z_0)} \circ h(z)$$

(where ψ_α as in subsection 2.1 & $\theta \in \mathbb{R}$ to be chosen)

is holo. injective, $f(\Omega) \subset \mathbb{D}$, and $f(z_0) = 0$.

Furthermore $f'(z_0) = e^{i\theta} \, \psi'_{h(z_0)}(h(z_0)) \, h'(z_0)$.

Hence if $\theta = -\arg\left(\psi'_{h(z_0)}(h(z_0))\, h'(z_0)\right)$,

then $f'(z_0) > 0$. ※

Step 2: The proof can be reduced to the case that Ω is a simply-connected region in $\mathbb{D}$ with $z_0 = 0 \in \Omega$.

Pf If Riemann Mapping Thm holds in the case as in Step 2, then $\exists$ conformal $f_1 : f(\Omega) \to \mathbb{D}$ conformal,

$f_1(0) = 0$ & $f_1'(0) > 0$,

where f is given in Step 1.

Then $F = f_1 \circ f : \Omega \to \mathbb{D}$ is conformal

and $F(z_0) = f_1(f(z_0)) = f_1(0) = 0$.

$F'(z_0) = f_1'(0) f'(z_0) > 0$

Step 3: For simply-connected region $\Omega \subset \mathbb{D}$ containing 0,

$\exists\, F \in \mathcal{F} = \{ f : \Omega \to \mathbb{D} : \text{holo., injective } \& \ f(0) = 0 \}$

s.t. $|F'(0)| = \sup_{f \in \mathcal{F}} |f'(0)|$

Pf: Clearly $f : \Omega^{CD} \to \mathbb{D} : z \mapsto z \in \mathcal{F}$

$\therefore \mathcal{F} \neq \emptyset$.

This also implies $S = \sup_{f \in \mathcal{F}} |f'(0)| \geq 1$.

On the other hand, by Cauchy inequality (Cor 4.3 in Ch 2)

$$S = \sup_{f \in \mathcal{F}} |f'(0)| < \infty \qquad \left(\text{since } f \in \mathcal{F} \Rightarrow |f| \leq 1\right)$$

Hence $\exists f_n \in \mathcal{F}$ such that

$$|f_n'(0)| \to S \quad \text{as } n \to \infty.$$

By Montel's Theorem (Thm 3.3), $\mathcal{F}$ is normal.

(as $\mathcal{F}$ is uniformly bounded)

$\Rightarrow \exists$ subseq (let call it f_n again)
converges uniformly on every compact subset to
a holo f on Ω.

Then $f(0) = 0$ and $|f'(0)| = S$

$S \geq 1 \Rightarrow f \not\equiv$ constant.

Hence Prop 3.5 $\Rightarrow$

f is injective, as f_n are injective.

$\therefore f \in \mathcal{F}$. This proves step 3.