Thm 2.4 $\zeta(s)$ has a meromorphic continuation into the whole $\mathbb{C}$ whose only singularity is a simple pole at $s=1$.

Pf: By definition of $\xi(s)$, we have

$$\zeta(s) = \pi^{\frac{s}{2}} \frac{\xi(s)}{\Gamma(\frac{s}{2})}$$

By Thm 1.6, $1/\Gamma(\frac{s}{2})$ is entire with simple zeros at

$s = 0, -2, -4, \cdots$

$\Rightarrow$ $s=0$ is a removable singularity of $\xi(s)/\Gamma(\frac{s}{2})$.

$\Rightarrow$ $\zeta(s)$ is meromorphic with a simple pole at $s=1$ only.

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Question: What is $\operatorname{res}_{s=1} \zeta(s)$? (Ex!)

Prop 2.5 $\exists$ seq. of entire function $\{\delta_n(s)\}_{n=1}^{\infty}$ such that

- $|\delta_n(s)| \leq \dfrac{|s|}{n^{\operatorname{Re}s+1}}$, $\forall s \in \mathbb{C}$ and

(8) — • $\displaystyle\sum_{1\leq n<N} \frac{1}{n^s} - \int_1^N \frac{dx}{x^s} = \sum_{1\leq n<N} \delta_n(s)$, $(N=2,3,\cdots)$.

Pf: Define $\displaystyle\delta_n(s) = \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s}\right) dx$ (entire)

By path integral of $f(z) = s z^{-s-1}$ (for $\mathrm{Re}\, z > 0$)

along $z(t) = x + t(n-x)$, $t \in [0,1]$, we have

$$\left| \frac{1}{n^s} - \frac{1}{x^s} \right| \le \int_0^1 |s| \left| [x + t(n-x)]^{-s-1} \right| |n-x| \, dt$$

$$\le \frac{|s|}{n^{\sigma+1}} \quad \text{for } x \in [n, n+1] \quad (\text{where } \sigma = \mathrm{Re}\, s)$$

$$\Rightarrow \quad |\delta_n(s)| \le \int_n^{n+1} \left| \frac{1}{n^s} - \frac{1}{x^s} \right| dx \le \frac{|s|}{n^{\sigma+1}}.$$

Summing up

$$\sum_{n=1}^{N-1} \delta_n(s) = \sum_{n=1}^{N-1} \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) dx = \sum_{n=1}^{N-1} \frac{1}{n^s} - \int_1^N \frac{dx}{x^s}. \quad \#$$

Cor 2.6 For $\mathrm{Re}(s) > 0$,

$$\zeta(s) - \frac{1}{s-1} = H(s)$$

where $H(s) = \sum_{n=1}^{\infty} \delta_n(s)$ is holo. in $\{\mathrm{Re}(s) > 0\}$.

Pf: For $\mathrm{Re}(s) > 1$,

the LHS of the formula (8) $\to \zeta(s) - \frac{1}{s-1}$ as $N \to \infty$.

For the RHS, Prop 2.5 $\Rightarrow$ $|\delta_n(s)| \le \frac{|s|}{n^{\mathrm{Re}(s)+1}}$, $\forall n$

$\Rightarrow \sum_{n=1}^{\infty} \delta_n(s)$ converges uniformly on

$\{|s|<R\} \cap \{Re(s)>0\}, \quad \forall R>0;$

in fact on $\{|s|<R\} \cap \{Re(s) \geq \delta\}, \forall R>0 \ \& \ \delta>0$
since $\sum \frac{1}{n^{\delta+1}} < \infty$ for $\delta>0$

Hence $H(s) = \sum_{n=1}^{\infty} \delta_n(s)$ is holo. on $\{Re(s)>0\} \supset \{Re(s)>1\}$

$\therefore \quad \zeta(s) - \frac{1}{s-1} = H(s)$ for $Re(s)>1$.

Note that $\zeta(s)$ has analytic continuation to $\{Re(s)>0\} \setminus \{s=1\}$

By uniqueness, the equality $\zeta(s) - \frac{1}{s-1} = H(s)$ also holds

on $\{Re(s)>0\}$.

$\#$

Prop 2.7 Suppose $s = \sigma + it$, $(\sigma, t \in \mathbb{R})$. Then $\forall \sigma_0 \in (0,1]$,
($\sigma_0 = 0$ is not needed)

and $\forall \varepsilon>0$, $\exists$ constant C_ε (depending on $\varepsilon>0$ only)

(i) $|\zeta(s)| \leq C_\varepsilon |t|^{1-\sigma_0+\varepsilon}$ for $\sigma_0 \leq \sigma$ & $|t| \geq 1$.

(ii) $|\zeta'(s)| \leq C_\varepsilon |t|^{\varepsilon}$ for $1 \leq \sigma$ & $|t| \geq 1$

Remark: In particular, one has

$$\begin{cases} \zeta(1+it) = O(|t|^{\varepsilon}) \\ \zeta'(1+it) = O(|t|^{\varepsilon}) \end{cases} \quad \text{as } |t| \to \infty.$$

Pf of Prop 2.7 :

Prop 2.5 $\Rightarrow$ $|\delta_n(s)| \leq \dfrac{|s|}{n^{\sigma+1}} \leq \dfrac{|s|}{n^{\sigma_0+1}}$

And $\delta_n(s) = \int_n^{n+1} \left(\dfrac{1}{n^s} - \dfrac{1}{x^s}\right) dx$

also

$\Rightarrow$ $|\delta_n(s)| \leq \left|\dfrac{1}{n^s}\right| + \left|\dfrac{1}{x^s}\right|$ for $x \in [n, n+1]$

$\leq \dfrac{2}{n^\sigma} \leq \dfrac{2}{n^{\sigma_0}}$

Then $\forall\ 0 \leq \delta \leq 1$

$$|\delta_n(s)| = |\delta_n(s)|^{\delta} |\delta_n(s)|^{1-\delta} \leq \left(\frac{|s|}{n^{\sigma_0+1}}\right)^{\delta} \left(\frac{2}{n^{\sigma_0}}\right)^{1-\delta}$$

$$\leq \frac{2|s|^{\delta}}{n^{\sigma_0+\delta}}$$

If $0 < \varepsilon \leq \sigma_0$, then $\delta = 1 - \sigma_0 + \varepsilon \leq 1$

$$\therefore \quad |\delta_n(s)| \leq \frac{2|s|^{1-\sigma_0+\varepsilon}}{n^{1+\varepsilon}} .$$

Cor 2.6

$\Rightarrow$ $$|\zeta(s)| \leq \frac{1}{|s-1|} + 2|s|^{1-\sigma_0+\varepsilon} \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}}$$

$\forall\ \sigma \geq \sigma_0$ and $|t| \geq 1$

For $\sigma \geq 2$, $$|\zeta(s)| \leq \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

For $0 < \sigma_0 \leq \sigma < 2$ and $|t| \geq 1$, $|s| = |t| \left|\frac{\sigma}{t} + i\right| \leq 3|t|$,

$$|\zeta(s)| \leq C + \left(2 \cdot 3^{1+\varepsilon} \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}}\right) |t|^{1-\sigma_0+\varepsilon} \leq C_\varepsilon |t|^{1-\sigma_0+\varepsilon}$$

Together $\Rightarrow$ for $0 < \sigma_0 \leq \sigma$ & $|t| \geq 1$, we have

$$|\zeta(s)| \lesssim C_\varepsilon |t|^{1-\sigma_0+\varepsilon} \qquad (\text{a new } C_\varepsilon)$$

In particular, choosing $\varepsilon' > 0$ sufficiently small, $\exists\ \delta' \leq 1$ s.t.

$$|\zeta(s)| \lesssim C_{\varepsilon'} |t|^{\delta'} \qquad \text{for } 0 < \sigma_0 \leq \sigma \ \& \ |t| \geq 1$$

Hence for $\varepsilon > \sigma_0$, $\delta = 1 - \sigma_0 + \varepsilon > 1 > \delta'$, and we have

$$|\zeta(s)| \leq C_{\varepsilon'} |t|^{\delta}, \ \forall\, 0 < \sigma_0 \leq \sigma \ \& \ |t| \geq 1$$

Therefore, we have proved that

$$\forall \varepsilon > 0, \qquad |\zeta(s)| \leq C_\varepsilon |t|^{1-\sigma_0+\varepsilon} \quad \text{on } 0 < \sigma_0 \leq \sigma \ \& \ |t| \geq 1$$

This proves (i).

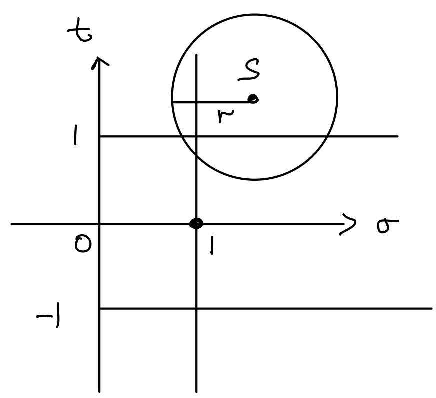
To prove (ii), for $\sigma \geq 1$ & $|t| \geq 1$,

the circle $\quad s + re^{i\theta}, \ \theta \in [0, 2\pi]$

with radius $r < 1$

lies in the half plane

$$\{\sigma + it : \sigma > 1 - r\}$$



Take $\sigma_0 = 1 - r$ & $\varepsilon = r$ in (i), we have

$$|\zeta(s + re^{i\theta})| \leq C_r |t + r\sin\theta|^{1-(1-r)+r}$$

$$\leq C_r' |t|^{2r} \qquad \text{for } |t| - r \geq 1.$$

If $|t| - r \leq 1 \Rightarrow |t| \lesssim 2$,

$\Rightarrow |\zeta(s+re^{i\theta})|$ is bounded (depending on r)

as $|s+re^{i\theta}-1| \geq |s-1| - r = 1-r$

Hence $|\zeta(s+re^{i\theta})| \leq C''_r |t|^{2r} \quad \forall\, |t| \geq 1$ (& $\sigma \geq 1$)

Then Cauchy integral famula

$$\Rightarrow \quad |\zeta'(s)| \leq \frac{1}{2\pi r} \int_0^{2\pi} |\zeta(s+re^{i\theta})| d\theta$$

$$\leq \frac{1}{r} C''_r |t|^{2r}, \quad \forall\, |t| \geq 1 \ \& \ \sigma \geq 1.$$

Since $1 > r > 0$ is arbitrary, we have that $\forall\, 0 < \varepsilon < 2$

$$|\zeta'(s)| \lesssim C_\varepsilon |t|^\varepsilon, \quad \forall\, |t| \geq 1 \ \& \ \sigma \geq 1$$

Using $|t| \geq 1$, we have $\forall\, \varepsilon \geq 2$,

$$|\zeta'(s)| \leq C_1 |t| \leq C_1 |t|^\varepsilon \quad \forall\, |t| \geq 1 \ \& \ \sigma \geq 1$$

Altogether, $\forall \varepsilon > 0$, $\exists\ C_\varepsilon > 0$ s.t.

$$|\zeta'(s)| \leq C_\varepsilon |t|^\varepsilon, \quad \forall |t| \geq 1 \ \& \ \sigma \geq 1$$

This proves (ii).