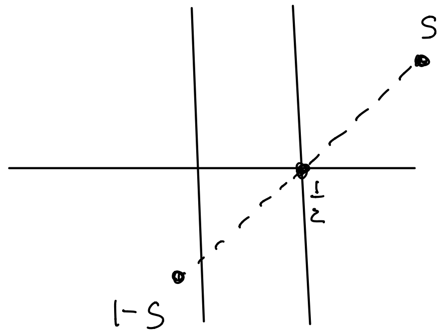
1.2 Further Properties of $\Gamma(s)$

> **Thm 1.4** $\forall s \in \mathbb{C} \setminus \mathbb{Z}$
>
> $$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s} \qquad \text{——— (4)}$$

Remark:

(1) $s \mapsto 1-s$

is the reflection across $\frac{1}{2}$



$\operatorname{Re} s > \frac{1}{2}$

(2) Thm 1.4 $\Rightarrow$ $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. (Since $\Gamma(s) > 0$, $\forall s > 0$)

Pf: Note that $\Gamma(s)$ mero. with simple poles at $s = 0, -1, -2, \cdots$

$\Rightarrow$ $\Gamma(1-s)$ mero. with simple poles at $s = 1, 2, 3, \cdots$

$\therefore$ $\Gamma(s)\Gamma(1-s)$ is mero. with simple poles at $s \in \mathbb{Z}$.

Clearly, $\frac{\pi}{\sin \pi z}$ is also mero. with simple poles at $s \in \mathbb{Z}$.

Therefore, by connectedness of $\mathbb{C} \setminus \mathbb{Z}$, it suffices to show that $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi z}$ on a subset of $\mathbb{C} \setminus \mathbb{Z}$ with accumulation points.

Note that for $0 < s < 1$ (a subset with accumulation points)

$$\Gamma(1-s) = \int_0^\infty e^{-u} u^{(1-s)-1} du$$ (changed the notation of the dummy variable)

$$= \int_0^\infty e^{-u} u^{-s} du$$

$$= \int_0^\infty e^{-tv} (tv)^{-s} \, t \, dv \qquad \forall t > 0$$

$$\Rightarrow \quad \Gamma(1-s)\Gamma(s) = \Gamma(1-s) \int_0^\infty e^{-t} t^{s-1} dt$$

$$= \int_0^\infty e^{-t} t^{s-1} \Gamma(1-s) \, dt$$

$$= \int_0^\infty e^{-t} t^{s} \left(\int_0^\infty e^{-tv} (tv)^{-s} dv \right) dt$$

$$= \int_0^\infty \int_0^\infty e^{-t(1+v)} v^{-s} dv \, dt$$

$$= \int_0^\infty \left(\int_0^\infty e^{-t(1+v)} dt \right) v^{-s} dv$$

(exponential decay at $t, v \to \infty$
and integrability of $\int_0^1 v^{-s} ds$ for $0 < s < 1$
$\Rightarrow$ integrals are absolutely integrable and hence
Fubini theorem applies.)

$$\therefore \quad \Gamma(1-s)\Gamma(s) = \int_0^\infty \frac{1}{1+v} v^{-s} dv$$

$$= \int_{-\infty}^\infty \frac{e^{(1-s)x}}{1+e^x} dx$$

(using $0<s<1$ $\Leftrightarrow$ $0<1-s<1$) $\qquad = \dfrac{\pi}{\sin \pi(1-s)}$ (by Eg 2 of §2.1 of Ch3 of the Text page 79)

$$= \frac{\pi}{\sin \pi s}$$

Therefore, uniqueness thm $\Rightarrow$ $\Gamma(1-s)\Gamma(s) = \dfrac{\pi}{\sin \pi s}$, $\forall s \in \mathbb{C} \setminus \mathbb{Z}$. #

Thm 1.6 (i) $1/\Gamma(s)$ is an entire function of s with simple zeros at $s=0,-1,-2,\cdots$ & $1/\Gamma(s) \neq 0$ for $s \in \mathbb{C}\setminus\{0,-1,-2,\cdots\}$.

(ii) $$\left|\frac{1}{\Gamma(s)}\right| \le C_1 e^{C_2|s|\log|s|},$$ for some constants $C_1, C_2 > 0$.

$\Rightarrow$ $1/\Gamma(s)$ is of <u>order 1</u>:

$\forall \varepsilon > 0$, $\exists\, C = C(\varepsilon) > 0$ s.t. $\left|\frac{1}{\Gamma(s)}\right| \le C(\varepsilon) e^{C_2|s|^{1+\varepsilon}}$.

<u>Pf</u>: By Thm 1.4, $$\frac{1}{\Gamma(s)} = \Gamma(1-s)\frac{\sin \pi s}{\pi}$$

Note that $\Gamma(1-s)$ has simple poles at $s=1,2,3,\cdots$ $(1-s=0,-1,-2,\cdots)$ and $\sin\pi s$ has simple zeros at $s=1,2,3,\cdots$

So $s=1,2,3,\cdots$ are removable singularities for $1/\Gamma(s)$.

Together with the fact the Γ has no other singularity,

$\frac{1}{\Gamma(s)}$ is entire,

and vanishing only at $s=0,-1,-2,\cdots$ (the simple poles of $\Gamma(s)$).

This proves (i).

To prove (ii), we are going to use formula (3) (alternative proof of Thm 1.3)

$$\Gamma(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\cdot\frac{1}{n+s} + \int_1^\infty e^{-t} t^{s-1}\,dt$$

which implies

$$\Gamma(1-s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{n+1-s} + \int_1^{\infty} e^{-t} t^{-s}\, dt$$

and hence

$$\frac{1}{\Gamma(s)} = \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{n+1-s}\right) \frac{\sin \pi s}{\pi} + \left(\int_1^{\infty} e^{-t} t^{-s}\, dt\right) \frac{\sin \pi s}{\pi}$$

For simplicity, let $\sigma = \operatorname{Re} s$.

Then $$\left|\int_1^{\infty} e^{-t} t^{-s}\, dt\right| \leq \int_1^{\infty} e^{-t} t^{-\sigma}\, dt \leq \int_1^{\infty} e^{-t} t^{|\sigma|}\, dt$$

Choose $n \in \mathbb{N}$ s.t. $|\sigma| \leq n \leq |\sigma| + 1$.

Then $$\int_1^{\infty} e^{-t} t^{|\sigma|}\, dt \leq \int_1^{\infty} e^{-t} t^{n}\, dt \leq \int_0^{\infty} e^{-t} t^{n}\, dt$$

$$= \Gamma(n+1) = n! \qquad \text{(Lemma 1.2)}$$

$$\leq n^n = e^{n \log n}$$

$$\leq e^{(|\sigma|+1)\log(|\sigma|+1)} \leq e^{(|s|+1)\log(|s|+1)}$$

We also have $|\sin \pi s| \leq e^{\pi |s|}$ (eg 1 of §2 of Ch5).

$\therefore$ The 2^{nd} term of $1/\Gamma(s)$ has bound

$$C e^{(|s|+1)\log(|s|+1)} \cdot e^{\pi |s|}$$

$$\leq C_1 e^{C_2 |s| \log |s|}$$ for some constants $C_1, C_2 > 0$. (Ex!)

For the 1^{st} term $\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{n+1-s}\right) \frac{\sin \pi s}{\pi}$:

<u>Case 1</u> $(|\operatorname{Im}(s)| > 1$

Then $|n+1-s| \geq |\mathrm{Im}\, s| > 1$,

$$\left| \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{n+1-s} \right| \leq \sum_{n=0}^{\infty} \frac{1}{n!} = e$$

$\therefore$ The 1st term $\leq C e^{\pi |s|}$ for some $C > 0$.

Case 2 $\quad |\mathrm{Im}(s)| \leq 1$.

Choose $k \in \mathbb{Z}$ s.t. $\quad k - \frac{1}{2} \leq \mathrm{Re}(s) < k + \frac{1}{2}$.

If $k \geq 1$,

$k = n+1$

$$\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{n+1-s} \right) \frac{\sin \pi s}{\pi} = \frac{(-1)^{k-1}}{(k-1)!} \cdot \frac{1}{k-s} \frac{\sin \pi s}{\pi}$$

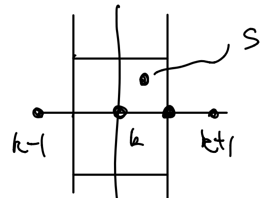
$$+ \sum_{n \neq k-1} \frac{(-1)^n}{n!} \frac{1}{n+1-s} \cdot \frac{\sin \pi s}{\pi}$$

Note $\left| \frac{(-1)^{k-1}}{(k-1)!} \frac{1}{k-s} \frac{\sin \pi s}{\pi} \right| \leq C$ since $\frac{\sin \pi s}{k-s}$ has removable

singularity at $s = k$ $\left(k - \frac{1}{2} \leq \mathrm{Re}\, s \leq k + \frac{1}{2} \;\; \& \;\; |\mathrm{Im}(s)| \leq 1 \right)$

(C is independent of k or s since $|\sin \pi s|$ is periodic)

and $\left| \sum_{n \neq k-1} \frac{(-1)^n}{n!} \frac{1}{n+1-s} \cdot \frac{\sin \pi s}{\pi} \right|$



$$\leq 2 \sum_{n=0}^{\infty} \frac{1}{n!} \frac{|\sin \pi s|}{\pi}$$

$|n+1-s| \geq \frac{1}{2}, \; \forall \; n \neq k-1$

$\leq C$

If $k \leq 0$, then $k - \frac{1}{2} \leq \operatorname{Re} s \leq k + \frac{1}{2} \Rightarrow \operatorname{Re} s \leq \frac{1}{2}$.

$$\Rightarrow \quad |n+1-s| \geq \frac{1}{2}, \quad \forall n = 0, 1, 2, \cdots$$

$$\Rightarrow \left| \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{n+1-s} \cdot \frac{\sin \pi s}{\pi} \right| \leq 2 \sum_{n=0}^{\infty} \frac{1}{n!} \frac{|\sin \pi s|}{\pi}$$

$$\leq C$$

since $|\operatorname{Im} s| \leq 1$ (& periodic of $\sin \pi s$).

All together

$$1/\Gamma(s) \leq C e^{\pi |s|} + C_1 e^{C_2 |s| \log |s|}$$

$$\Rightarrow \quad \frac{1}{\Gamma(s)} \leq C_1 e^{C_2 |s| \log |s|} \quad (\text{maybe for new } C_1, C_2 > 0)$$

This proves the 1st statement of (ii).

The 2nd statement of (ii) follows from $|s| \log |s| \leq C |s|^{1+\varepsilon}, \forall \varepsilon > 0$,

and $\sum \frac{1}{n^\sigma}$ converges $\Leftrightarrow \sigma > 1$,

and Thm 2.1 of §2 of Ch5. (Ex!)

#

Thm 1.7 $\forall s \in \mathbb{C}$

$$\frac{1}{\Gamma(s)} = e^{\gamma s} s \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-\frac{s}{n}},$$

where $\gamma = \lim_{N \to \infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \log N \right)$ is the Euler's const.

Pf: By Hadamard factorization theorem (Thm 5.1 in §5 of Ch5) & Thm 1.6,

$$\frac{1}{\Gamma(s)} = e^{As+B}\, s \prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-\frac{s}{n}}$$

By remark (1) after the proof of Thm 1.3, $\lim_{s\to 0} s\Gamma(s) = 1$.

$$\Rightarrow \quad 1 = e^{B}\quad \text{, ie } B=0 \ (\text{or } B = 2\pi i k,\ k\in\mathbb{Z})$$

$$\therefore \quad \frac{1}{\Gamma(s)} = e^{As}\, s \prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-\frac{s}{n}}$$

Putting $s=1$, we have

$$1 = \frac{1}{\Gamma(1)} = e^{A} \prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)e^{-\frac{1}{n}}$$

$$\Rightarrow \quad e^{-A} = \lim_{N\to\infty}\prod_{n=1}^{N}\left(1+\frac{1}{n}\right)e^{-\frac{1}{n}} = \lim_{N\to\infty}\prod_{n=1}^{N} e^{\log\left(1+\frac{1}{n}\right)-\frac{1}{n}}$$

$\Rightarrow$ for some $k\in\mathbb{Z}$,

$$-A + 2\pi i k = \lim_{N\to\infty}\sum_{n=1}^{N}\left[\log\left(\frac{n+1}{n}\right) - \frac{1}{n}\right]$$

$$= \lim_{N\to\infty}\left(\log\frac{2}{1} + \log\frac{3}{2} + \cdots + \log\frac{N}{N-1} + \log\frac{N+1}{N}\right) - \sum_{n=1}^{N}\frac{1}{n}$$

$$= -\lim_{N\to\infty}\left[\left(\sum_{n=1}^{N}\frac{1}{n} - \log N\right) - \log\left(1+\frac{1}{N}\right)\right]$$

$$= -\gamma$$

$$\therefore \quad \frac{1}{\Gamma(s)} = e^{(\gamma - 2\pi i k)s}\cdot s\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-\frac{s}{n}}, \qquad \forall s\in\mathbb{C}.$$

Putting $s=\frac{1}{2}$, (in fact, real not integer) we have $k=0$. ※

§2 The Zeta Function

Def: The Riemann Zeta Function for $s>1$ ($s\in\mathbb{R}$) is defined by
$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}.$$

2.1 Functional Equation & Analytic Continuation

Prop 2.1 $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ converges for $Re(s)>1$, and holomorphic in $\{s: Re(s)>1\}$

Pf: Let $s=\sigma+it$ $(\sigma,t\in\mathbb{R})$.

Then $\left|\frac{1}{n^s}\right| = e^{Re(-s\log n)} = e^{-\sigma\log n} = \frac{1}{n^\sigma}$

$\Rightarrow$ $\forall\,\delta>0$, then for $\sigma>1+\delta$,

$$\left|\frac{1}{n^s}\right| \le \frac{1}{n^{1+\delta}} \text{ and } \sum_{n=1}^{\infty}\frac{1}{n^{1+\delta}}<+\infty \Rightarrow$$

the series $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ converges absolutely uniformly and hence define a holo. function on $\{s=\sigma+it : \sigma>1+\delta\}$.

Since $\delta>0$ is arbitrary, $\zeta(s)$ is defined and holomorphic on $\{s=\sigma+it : \sigma>1\}$. ※

Recall: The Theta Function defined for $t>0$ by

$$\vartheta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}$$

satisfies $\boxed{\vartheta(t) = t^{-\frac{1}{2}}\,\vartheta\left(\frac{1}{t}\right)}$ (Application (1) of Poisson summation formula) (Thm 2.4 in Ch 4)

Note that $\vartheta(t) - 1 = 2\sum_{n=1}^{\infty} e^{-\pi n^2 t}$

$$\Rightarrow \quad |\vartheta(t)-1| < 2\sum_{n=1}^{\infty}\left(e^{-\pi t}\right)^n = \frac{2e^{-\pi t}}{1-e^{-\pi t}} \quad (\text{for } t>0)$$

Hence $\exists\, C>0$ s.t.

$$|\vartheta(t)-1| \le Ce^{-\pi t} \quad \text{for } t \ge 1.$$

Then
$$\begin{aligned}\vartheta(t) &\le t^{-\frac{1}{2}}\left(1+Ce^{-\frac{\pi}{t}}\right) \quad \text{for } t<1\\ &\le Ct^{-\frac{1}{2}} \quad \text{as } t\to 0.\end{aligned}$$

In summary
$$\boxed{\begin{cases} \vartheta(t) \le Ct^{-\frac{1}{2}} & \text{as } t\to 0 \\ |\vartheta(t)-1| \le Ce^{-\pi t} & \text{as } t\to\infty \end{cases}}$$

Thm 2.2 If $\mathrm{Re}(s)>1$, then
$$\pi^{-\frac{s}{2}}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{2}\int_0^{\infty} u^{\frac{s}{2}-1}\left(\vartheta(u)-1\right)du$$

Pf: By properties summarized above, for $\mathrm{Re}\, s>1$
$$\left|u^{\frac{s}{2}-1}\left(\vartheta(u)-1\right)\right| \le u^{\frac{\mathrm{Re}\, s}{2}-1}\left|\vartheta(u)-1\right|$$

$$\leq \begin{cases} C\, u^{\frac{\operatorname{Re} s-1}{2}-1} & \text{as } u \to 0 \\ C\, u^{\frac{\operatorname{Re} s}{2}-1} e^{-\pi u} & \text{as } u \to \infty \end{cases}$$

$\therefore \; \frac{1}{2}\int_0^\infty u^{\frac{s}{2}-1}(\vartheta(u)-1)\,du$ converges absolutely

and hence $\displaystyle = \int_0^\infty u^{\frac{s}{2}-1}\left(\frac{\vartheta(u)-1}{2}\right)du$

$$= \int_0^\infty u^{\frac{s}{2}-1}\left(\sum_{n=1}^\infty e^{-\pi n^2 u}\right)du$$

$$= \sum_{n=1}^\infty \int_0^\infty u^{\frac{s}{2}-1} e^{-\pi n^2 u}\,du$$

(change of variable $u = \frac{t}{\pi n^2}$) $\displaystyle = \sum_{n=1}^\infty \int_0^\infty \left(\frac{t}{\pi n^2}\right)^{\frac{s}{2}-1} e^{-t} \cdot \frac{dt}{\pi n^2}$

$$= \sum_{n=1}^\infty \frac{1}{(\pi n^2)^{s/2}} \int_0^\infty e^{-t}\, t^{\frac{s}{2}-1}\,dt$$

$$= \pi^{-\frac{s}{2}}\,\Gamma\!\left(\frac{s}{2}\right) \sum_{n=1}^\infty \frac{1}{n^s}$$

$$= \pi^{-\frac{s}{2}}\,\Gamma\!\left(\frac{s}{2}\right)\zeta(s)$$

#

Def The Xi Function $\xi(s)$ is defined by

$$\xi(s) = \pi^{-\frac{s}{2}}\,\Gamma\!\left(\frac{s}{2}\right)\zeta(s)$$

Thm 2.3

- $\xi(s)$ is holo. for $\mathrm{Re}(s)>1$
- $\xi(s)$ has analytic continuation as a meromorphic function to $\mathbb{C}$ with simple poles at $s=0$ & $s=1$. (with $\mathrm{res}_{s=0}\,\xi(s)=-1$, $\mathrm{res}_{s=1}\,\xi(s)=1$)
- And $$\xi(s)=\xi(1-s),\ \forall s\in\mathbb{C}\setminus\{0,1\}$$

Pf: To simplify notation, let $\psi(u)=\frac{1}{2}(\vartheta(u)-1)$.

Then $\vartheta(u)=1+2\psi(u)$

$\Rightarrow$

$$1+2\psi(u)=\vartheta(u)=u^{-\frac{1}{2}}\vartheta\left(\tfrac{1}{u}\right)$$
$$=u^{-\frac{1}{2}}\left(1+2\psi\left(\tfrac{1}{u}\right)\right)$$

$\therefore$ $$\psi(u)=u^{-\frac{1}{2}}\psi\left(\tfrac{1}{u}\right)+\frac{1}{2u^{\frac{1}{2}}}-\frac{1}{2},\quad \forall u\in(0,\infty)$$

By Thm 2.2, for $\mathrm{Re}\,s>1$, we have

$$\xi(s)=\int_0^\infty u^{\frac{s}{2}-1}\psi(u)\,du$$
$$=\int_0^1 u^{\frac{s}{2}-1}\left[u^{-\frac{1}{2}}\psi\left(\tfrac{1}{u}\right)+\frac{1}{2u^{\frac{1}{2}}}-\frac{1}{2}\right]du+\int_1^\infty u^{\frac{s}{2}-1}\psi(u)\,du$$
$$=\frac{1}{2}\int_0^1 u^{\frac{s-1}{2}-1}\,du-\frac{1}{2}\int_0^1 u^{\frac{s}{2}-1}\,du$$
$$+\int_0^1 u^{\frac{s}{2}-\frac{3}{2}}\psi\left(\tfrac{1}{u}\right)du+\int_1^\infty u^{\frac{s}{2}-1}\psi(u)\,du$$
$$=\frac{1}{2}\left[\frac{u^{\frac{s-1}{2}}}{\frac{s-1}{2}}\right]_0^1-\frac{1}{2}\left[\frac{u^{\frac{s}{2}}}{\frac{s}{2}}\right]_0^1+\int_\infty^1 u^{-\frac{s}{2}+\frac{3}{2}}\psi(u)\left(-\frac{du}{u^2}\right)$$
$$+\int_1^\infty u^{\frac{s}{2}-1}\psi(u)\,du$$

$$\therefore \quad \xi(s) = \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \left(u^{-\frac{s}{2}-\frac{1}{2}} + u^{\frac{s}{2}-1}\right)\psi(u)\,du \quad \text{—} (*)$$

Note that $|\psi(u)| = \frac{1}{2}|\vartheta(u)-1| \leq Ce^{-\pi u}$ as $u \to +\infty$

$\therefore$ $\int_1^\infty \left(u^{-\frac{s}{2}-\frac{1}{2}} + u^{\frac{s}{2}-1}\right)\psi(u)\,du$ converges absolutely for all $s \in \mathbb{C}$ (not just $\mathrm{Re}\, s > 1$) and defines an entire function. Hence the RHS of $(*)$ is a meromorphic function with simple poles at $s=0$ & $s=1$ (with corresponding residues). This proves the first two statements.

For the last formula, substitute $1-s$ in $(*)$, we have

$$\xi(1-s) = \frac{1}{(1-s)-1} - \frac{1}{(1-s)} + \int_1^\infty \left(u^{-\frac{1-s}{2}-\frac{1}{2}} + u^{\frac{1-s}{2}-1}\right)\psi(u)\,du$$

$$= -\frac{1}{s} + \frac{1}{s-1} + \int_1^\infty \left(u^{\frac{s}{2}-1} + u^{-\frac{s}{2}-\frac{1}{2}}\right)\psi(u)\,du$$

$$= \xi(s).$$

※