§5 Hadamard's Factorization Theorem

Thm 5.1 Suppose f entire, ρ_f = order of growth of f.
Let $k \in \mathbb{N}$ such that $k \le \rho_f < k+1$.
If $a_1, a_2 \cdots$ are the non-zero zeros of f, then

$$f(z) = e^{P(z)} z^m \prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right)$$

where P is a polynomial of $\deg P \le k$, and
m = order of zero of f at $z=0$ (could have $m=0$)

Remark: Recall that $E_k(z) = (1-z)e^{z+\frac{z^2}{2}+\cdots+\frac{z^k}{k}}$.
The different in Weierstrass & Hadamard is that k is fixed in Hadamard, independent of n, in $E_k\left(\frac{z}{a_n}\right)$, degree of the poly in the exponential in the factor $= k$.
But in Weierstrass, it is $E_n\left(\frac{z}{a_n}\right)$, the poly in the exponential in the factor has degree $\to +\infty$.

To prove Hadamard's Theorem, we start with some lemmas.
Conditions and notations as in the Thm.

Lemma 5.2:
$$|E_k(z)| \ge \begin{cases} e^{-c|z|^{k+1}} & \text{if } |z| \le \frac{1}{2} \\ |1-z|e^{-c'|z|^k} & \text{if } |z| \ge \frac{1}{2} \end{cases}$$
for some constants c & $c' > 0$. (c' depends on k)

Pf: If $|z| \leq \frac{1}{2}$, $\log(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}$ holds and hence

$$E_k(z) = (1-z)\, e^{z + \frac{z}{2} + \cdots + \frac{z^k}{k}}$$
$$= e^{-\sum_{n=k+1}^{\infty} \frac{z^n}{n}}$$

Let $w = -\sum_{n=k+1}^{\infty} \frac{z^n}{n}$ again, we have as in the proof of Weierstrass' Thm, we have

$$|w| \leq C|z|^{k+1} \quad \text{for some } C>0.$$

Hence $|E_k(z)| = |e^w| = e^{\operatorname{Re} w} \geq e^{-|w|} \geq e^{-C|z|^{k+1}}$

If $|z| > \frac{1}{2}$, then

$$|E_k(z)| = |1-z|\left|e^{z+\cdots+\frac{z^k}{k}}\right|$$
$$\geq |1-z|\, e^{-\left|z+\cdots+\frac{z^k}{k}\right|}$$
$$\geq |1-z|\, e^{-C'|z|^k} \quad \text{for some } C' \text{ since } |z| > \tfrac{1}{2}.$$

(depending on k)

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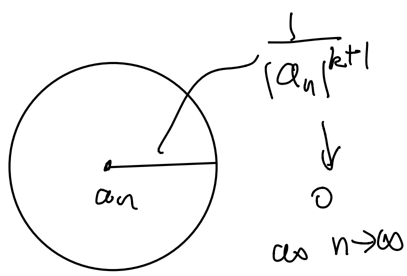
Lemma 5.3 $\forall\, s$ s.t. $\rho_f < s < k+1$, $\exists$ const. $C>0$ s.t

$$\left|\prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right)\right| \geq e^{-C|z|^s}$$

for $z \in \mathbb{C} \setminus \bigcup_{n=1}^{\infty} B_{\frac{1}{|a_n|^{k+1}}}(a_n)$.



(In the following, C means a constant indep. of z, may be different in each step.)

Pf

Step 1 : For any $z \in \mathbb{C}$,

$$\left| \prod_{|a_n| > 2|z|} E_k\left(\frac{z}{a_n}\right) \right| \geq e^{-c|z|^s} \quad \text{for some } c > 0.$$

Pf of Step 1:

(Convergence will be proved later)

$$\left| \prod_{|a_n|>2|z|} E_k\left(\frac{z}{a_n}\right) \right| = \prod_{|a_n|>2|z|} \left| E_k\left(\frac{z}{a_n}\right) \right|$$

$$\geq \prod_{|a_n|>2|z|} e^{-c\left|\frac{z}{a_n}\right|^{k+1}} \qquad \text{by Lemma 5.2 } \& \left|\frac{z}{a_n}\right| < \frac{1}{2}$$

$$= e^{-c|z|^{k+1} \sum_{|a_n|>2|z|} \frac{1}{|a_n|^{k+1}}}$$

$$s < k+1 \Rightarrow \frac{1}{|a_n|^{k+1}} = \frac{1}{|a_n|^s |a_n|^{k+1-s}} \leq \frac{1}{|a_n|^s} \cdot \frac{1}{2^{k+1-s} |z|^{k+1-s}}$$

$$\rho_f < s \Rightarrow (\text{by Thm 2.1 of Ch5}) \quad \sum \frac{1}{|a_n|^s} < +\infty.$$

$$\therefore \quad \sum_{|a_n|>2|z|} \frac{1}{|a_n|^{k+1}} \leq c \frac{1}{|z|^{k+1-s}}$$

(note this c is not the same c above)

Hence $$\left| \prod_{|a_n|>2|z|} E_k\left(\frac{z}{a_n}\right) \right| \geq e^{-c|z|^s}$$

(note this c is the product of the 2 different c's above)

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Step 2 For $z \in \mathbb{C} \setminus \bigcup_{n=1}^{\infty} B_{\frac{1}{|a_n|^{k+1}}}(a_n)$ and $|z| \geq \frac{1}{2} \min |a_n|$

$$\prod_{|a_n| \leq 2|z|} \left| 1 - \frac{z}{a_n} \right| \geq e^{-c|z|^s} \quad \text{for some } c > 0.$$

Pf of Step 2 :

For $z \in \mathbb{C} \setminus \bigcup_{n=1}^{\infty} B_{\frac{1}{|a_n|^{k+1}}}(a_n)$,

$$|z - a_n| \geq \frac{1}{|a_n|^{k+1}}$$

$$\Rightarrow \quad \prod_{|a_n| \leq 2|z|} \left|1 - \frac{z}{a_n}\right| \geq \prod_{|a_n| \leq 2|z|} \frac{1}{|a_n|^{k+2}}$$

$$= e^{-(k+2)\sum_{|a_n| \leq 2|z|} \log|a_n|}$$

Note that $(k+2)\sum_{|a_n| \leq 2|z|} \log|a_n| \leq (k+2) \sum_{|a_n| \leq 2|z|} \log(2|z|)$

$$\leq (k+2)\log(2|z|)\ n(2(1+\delta)|z|) \quad (\text{for any } \delta > 0)$$

Then for any s_1 s.t. $\rho_f < s_1 < s$, we have (by Thm 2.1)

$$n(2(1+\delta)|z|) \leq C\,(2(1+\delta)|z|)^{s_1}$$
$$\leq C|z|^{s_1} \quad (\text{for some } C > 0)$$

Hence

$$(k+2)\sum_{|a_n| \leq 2|z|} \log|a_n| \leq (k+2) \cdot \log(2|z|) \cdot C|z|^{s_1}$$
$$\leq C|z|^{s} \quad \text{since } s > s_1 \text{ and}$$
$$|z| \geq \tfrac{1}{2} \min|a_n|.$$

$$\therefore \quad \prod_{|a_n| \leq 2|z|} \left|1 - \frac{z}{a_n}\right| \geq e^{-(k+2)\sum_{|a_n| \leq 2|z|} \log|a_n|}$$

$$\geq e^{-C|z|^{s}}, \quad \text{for some } C > 0.$$

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Step 3 : For $z \in \mathbb{C} \setminus \bigcup_{n=1}^{\infty} B_{\frac{1}{|a_n|^{k+1}}}(a_n)$, and $|z| \geq \frac{1}{2} \min |a_n|$,

$$\left| \prod_{|a_n| \leq 2|z|} E_k\left(\frac{z}{a_n}\right) \right| \geq e^{-c|z|^s} \qquad \text{for some } c > 0.$$

Pf of Step 3 : By lemma 5.2

$$\left| \prod_{|a_n| \leq 2|z|} E_k\left(\frac{z}{a_n}\right) \right| = \prod_{|a_n| \leq 2|z|} \left| E_k\left(\frac{z}{a_n}\right) \right|$$

$$\geq \prod_{|a_n| \leq 2|z|} \left(\left|1 - \frac{z}{a_n}\right| e^{-c' \left|\frac{z}{a_n}\right|^k} \right)$$

$$= \left(\prod_{|a_n| \leq 2|z|} \left|1 - \frac{z}{a_n}\right| \right) \left(\prod_{|a_n| \leq 2|z|} e^{-c' \left|\frac{z}{a_n}\right|^k} \right)$$

$$(\text{by Step 2}) \quad \geq e^{-c|z|^s} \, e^{-c'|z|^k \sum_{|a_n| \leq 2|z|} \frac{1}{|a_n|^k}}$$

since $k \leq \rho_f < s$,

$$|a_n|^k = |a_n|^s |a_n|^{k-s} \leq |a_n|^s (2|z|)^{k-s}$$

$$\Rightarrow \sum_{|a_n| \leq 2|z|} \frac{1}{|a_n|^k} \leq c|z|^{s-k} \sum \frac{1}{|a_n|^s} \leq C|z|^{s-k}$$ (C's are different from each others)

Hence
$$\left| \prod_{|a_n| \leq 2|z|} E_k\left(\frac{z}{a_n}\right) \right| \geq e^{-c|z|^s} e^{-c|z|^k |z|^{s-k}}$$

$$= e^{-c|z|^s} \qquad \text{for some } c > 0.$$

Step 4 : Complete the proof of the lemma 5.3.

$\forall z \in \mathbb{C} \setminus B_{\frac{1}{|a_n|^{k+1}}}(a_n)$

If $|z| < \frac{1}{2} \min |a_n|$, then

$$\prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right) = \prod_{|a_n| > 2|z|} E_k\left(\frac{z}{a_n}\right).$$

Then Step 1 $\Rightarrow$ $\left|\prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right)\right| \gtrsim e^{-C|z|^s}$ for some $C>0$.

If $|z| \geq \frac{1}{2} \min |a_n|$, then

$$\prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right) = \prod_{|a_n| \leq 2|z|} E_k\left(\frac{z}{a_n}\right) \cdot \prod_{|a_n| > 2|z|} E_k\left(\frac{z}{a_n}\right)$$

Steps 1 & 3 $\Rightarrow$ $\left|\prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right)\right| \gtrsim e^{-C|z|^s} e^{-c|z|^s}$

$= e^{-C|z|^s}$ (all c are diff)

⋇

<u>Cor 5.4</u> $\exists$ a sequence $\{r_m\}$ with $r_m \to +\infty$ as $m \to +\infty$

(may choose $\{r_m\}$ to be increasing)

such that

$$\left|\prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right)\right| \gtrsim e^{-C|z|^s} \quad \text{for } |z| = r_m$$

for some constant $C>0$.

<u>Pf</u> Since $k+1 > \rho_f$, $\sum_n \frac{1}{|a_n|^{k+1}} < +\infty$,

$\exists$ $N > 0$ s.t. $\sum_{n=N}^{\infty} \frac{1}{|a_n|^{k+1}} < \frac{1}{10}$.

Consider intervals $I_n = \left[|a_n| - \frac{1}{|a_n|^{k+1}}, |a_n| + \frac{1}{|a_n|^{k+1}} \right]$.

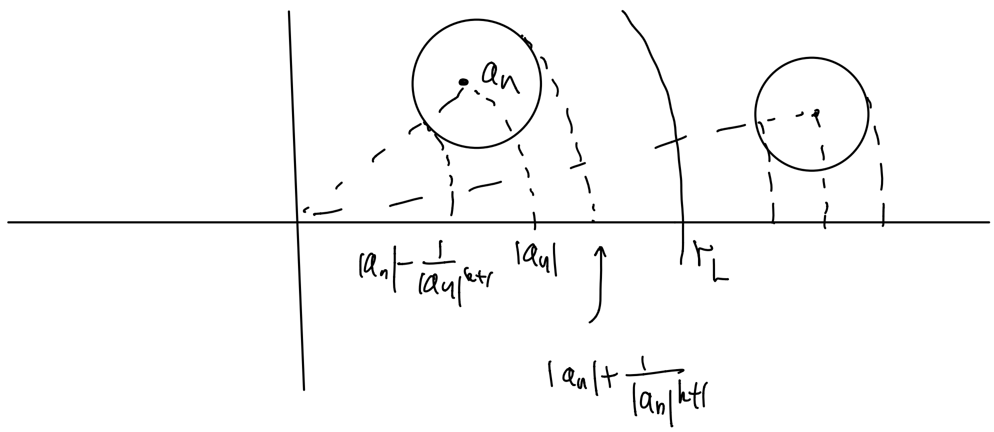
Then $|I_n| = \frac{2}{|a_n|^{k+1}}$.

$$\Rightarrow \quad \sum_{n=N}^{\infty} |I_n| = 2 \sum_{n=N}^{\infty} \frac{1}{|a_n|^{k+1}} < \frac{1}{5}.$$

Hence $\bigcup_{n=N}^{\infty} I_n$ cannot cover an interval of length 1.

$\Rightarrow \bigcup_{n=N}^{\infty} I_n$ cannot cover the interval $[L, L+1]$ (for sufficiently large $L \in \mathbb{N}$).

$\Rightarrow \exists\, r_L$ s.t. $|z| = r_L \Rightarrow z \notin B_{\frac{1}{|a_n|^{k+1}}}(a_n)$, $\forall n \geq 1$



By Lemma 5.3 $\left| \prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right) \right| \geq e^{-C|z|^s}$, $\forall |z| = r_L$,
($L \in \mathbb{N}$ sufficiently large)

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Lemma 5.5 Suppose g is entire, if $\exists$ seq. $\{r_m\}$, $r_m \to +\infty$ s.t.

$$\operatorname{Re} g(z) \leq C r_m^s \quad \text{for } |z| = r_m, \ \forall m \geq 1.$$

Then g is a polynomial of degree $\leq s$.

Pf: g entire $\Rightarrow$ $g(z) = \sum_{n=0}^{\infty} b_n z^n$, $\forall z \in \mathbb{C}$

By Cauchy integral famula (Fourier coefficients),

we have
$$\frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta}) e^{-in\theta} d\theta = \begin{cases} b_n r^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$\Rightarrow$ For $n > 0$, $\frac{1}{2\pi}\int_0^{2\pi} \overline{g(re^{i\theta})} e^{-in\theta} d\theta = 0$

Hence
$$\frac{1}{2\pi}\int_0^{2\pi} (g + \bar{g})(re^{i\theta}) e^{-in\theta} d\theta = b_n r^n, \quad n > 0$$

ie.
$$\int_0^{2\pi} [\operatorname{Re} g(re^{i\theta})] \cdot e^{-in\theta} d\theta = \pi b_n r^n, \quad \forall n > 0$$

For $n = 0$,
$$\int_0^{2\pi} \operatorname{Re} g(re^{i\theta}) d\theta = 2\pi \operatorname{Re}(b_0).$$

Note that
$$\int_0^{2\pi} e^{-in\theta} d\theta = 0, \ \forall n > 0$$

we have
$$b_n = \frac{1}{\pi r^n}\int_0^{2\pi} [\operatorname{Re} g(re^{i\theta}) - C r^s] e^{-in\theta} d\theta$$

$\Rightarrow$ for $r = r_m$,

$$|b_n| \leq \frac{1}{\pi r_m^n}\int_0^{2\pi} [C r_m^s - \operatorname{Re} g(r_m e^{i\theta})] d\theta$$

$$= \frac{2C}{\pi r_m^{n-s}} - \frac{2\operatorname{Re}(b_0)}{\pi r_m^n} \longrightarrow 0 \quad \text{as } r_m \to +\infty \text{ if } n > s$$

$\therefore$ g = poly. of degree $\leq s$. #

Pf of Hadamard's Theorem

Let $E(z) = z^m \prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right)$

Lemma 4.2 $\Rightarrow$ $\left|1 - E_k\left(\frac{z}{a_n}\right)\right| \leq c\left|\frac{z}{a_n}\right|^{k+1}$ for some c

Thm 2.1 $\Rightarrow$ $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{k+1}} < +\infty$ since $k+1 > \rho_f$

we have $\sum_{n=1}^{\infty} \left|1 - E_k\left(\frac{z}{a_n}\right)\right| \leq C|z|^{k+1}$ $C > 0$ indep. of z.

Hence the infinite product converges uniformly on $\{|z| < R\}$, $\forall R > 0$,
this implies $E(z)$ is a well-defined entire function.
Since $E(z)$ has the same zeros as $f(z)$,
$\frac{f(z)}{E(z)}$ is holomorphic and nowhere vanishing.

$$\Rightarrow \quad \frac{f(z)}{E(z)} = e^{g(z)} \quad \text{for some entire } g(z).$$

By Cor 5.4. for $|z| = r_m$,

$$e^{\operatorname{Re} g(z)} = \left|\frac{f(z)}{E(z)}\right| \leq \frac{Ae^{B|z|^s}}{e^{-c|z|^s}} \quad \forall\, s > \rho_f$$

$$= Ae^{(B+C)|z|^s}$$

$\Rightarrow$ $\forall\, |z| = r_m$, $\operatorname{Re} g(z) \leq C|z|^s$. (diff. C.)
By Lemma 5.5, $g(z) =$ polynomial of degree $\leq s$.

$\Rightarrow$ $g(z) =$ polynomial of degree $\leq k$. #

Ch6 The Gamma and Zeta Functions

§1 The Gamma Function

Def: For $s>0$, the gamma function is defined by

$$\Gamma(s) = \int_0^\infty e^{-t}\, t^{s-1}\, dt. \qquad \text{——— (1)}$$

Remark: Clearly $\Gamma(s)$ is well-defined for $s>0$:

$$\int_\varepsilon^1 t^{s-1}\, dt = \left.\frac{t^s}{s}\right|_\varepsilon^1 \to \frac{1}{s} \quad \text{as } \varepsilon \to 0.$$

(and e^{-t} bdd near $t=0$ & rapidly decay as $t \to +\infty$)

Prop 1.1 The formula $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$ extends the domain of definition of $\Gamma(s)$ to the open half-plane $\{s \in \mathbb{C} : \mathrm{Re}(s) > 0\}$.

Remark: By definition, $\int_0^\infty e^{-t} t^{s-1} dt = \lim_{\varepsilon \to 0} \int_\varepsilon^{\frac{1}{\varepsilon}} e^{-t} t^{s-1} dt$

Pf: It suffices to show that (1) defines a holomorphic function in $\quad S_{\delta,M} = \{\delta < \mathrm{Re}(s) < M\}, \ \forall\, 0<\delta<M<\infty.$

Note that $t^{s-1} = e^{(s-1)\log t}$ is holomorphic in $s \in \mathbb{C}$

and hence $e^{-t} t^{s-1}$ is holo. in s, and continuous in $(t,s) \in [\varepsilon, \frac{1}{\varepsilon}] \times [\delta, M]$. Hence Thm 5.4 of Ch 2 $\Rightarrow$

$\forall \varepsilon > 0$,

$$F_\varepsilon(s) = \int_\varepsilon^{\frac{1}{\varepsilon}} e^{-t} t^{s-1} dt$$

is holo. on $S_{\delta, M}$.

Note also that $\delta < Re(s) < M$

$$\Rightarrow \quad |e^{-t} t^{s-1}| = e^{-t} |e^{(s-1)\log t}| = e^{-t} e^{(Re(s)-1)\log t}$$

$$= e^{-t} t^{Re(s)-1}$$

(for $\varepsilon < 1$)

$$\Rightarrow \quad \int_0^\varepsilon |e^{-t} t^{s-1}| dt = \int_0^\varepsilon e^{-t} t^{Re(s)-1} dt$$

$$\leq \int_0^\varepsilon e^{-t} t^{\delta-1} dt \leq \frac{\varepsilon^\delta}{\delta}$$

$\therefore$ $\int_0^\varepsilon e^{-t} t^{s-1} dt$ is convergent. (as improper integral near 0)

Similarly $\int_{\frac{1}{\varepsilon}}^\infty |e^{-t} t^{s-1}| dt \leq \int_{\frac{1}{\varepsilon}}^\infty e^{-t} t^{M-1} dt$

$$\leq C \int_{\frac{1}{\varepsilon}}^\infty e^{-\frac{t}{2}} dt \quad (\text{for some } C > 0)$$

$$\leq 2C e^{-\frac{1}{2\varepsilon}}$$

$\therefore$ $\int_{\frac{1}{\varepsilon}}^\infty e^{-t} t^{s-1} dt$ is also convergent.

Hence $\int_0^\infty e^{-t} t^{s-1} dt$ defined &

$$\int_0^\infty e^{-t} t^{s-1} dt - F_\varepsilon(s) = \int_0^\varepsilon e^{-t} t^{s-1} dt + \int_{\frac{1}{\varepsilon}}^\infty e^{-t} t^{s-1} dt$$

$$\Rightarrow \left| \int_0^\infty e^{-t} t^{s-1} dt - F_\varepsilon(s) \right| \leq \int_0^\varepsilon e^{-t} t^{Re(s)-1} dt + \int_{\frac{1}{\varepsilon}}^\infty e^{-t} t^{Re(s)-1} dt$$

$$\leq \frac{\varepsilon^\delta}{\delta} + 2Ce^{-\frac{1}{2\varepsilon}} \to 0 \quad \text{as } \varepsilon \to 0.$$

$\therefore$ (Thm 5.2 of Ch2) $\int_0^\infty e^{-t} t^{s-1} dt$ is a uniform limit of sequence of holo. function on $S_{\delta,M}$, hence

$$\int_0^\infty e^{-t} t^{s-1} dt \text{ is a holo. on } S_{\delta,M}, \ \forall\, 0<\delta<M<\infty$$

$\Rightarrow$ $\int_0^\infty e^{-t} t^{s-1} dt$ define a holo on $\{Re(s)>0\}$.

Clearly, when restricted to real $s>0$, $\int_0^\infty e^{-t} t^{s-1} dt = \Gamma(s)$,

$\therefore$ $\int_0^\infty e^{-t} t^{s-1} dt$ is the required extension. ※