§2 Functions of Finite Order

Def: Let f be an entire function. If $\exists\, \rho > 0$ s.t. for some constants $A, B > 0$,

$$|f(z)| \le A e^{B|z|^\rho} \quad \forall z \in \mathbb{C},$$

then we say that f has an order of growth $\le \rho$.

And define the order of growth of f as

$$\rho_f = \inf\{\rho : f \text{ has an order of growth} \le \rho\},$$

eg: The order of growth of e^{z^2} is 2. (Ex!)

Remarks:

- Clearly, if f has an order of growth $\le \rho_1$ and $\rho_1 < \rho_2$, then f has an order of growth $\le \rho_2$. (Ex!)

- It is easy to see that for $f(z) = e^{z^2}$, $(\rho_f = 2)$ $\exists\, A, B > 0$ s.t.

$$|f(z)| \le A e^{B|z|^{\rho_f}}, \quad \forall z \in \mathbb{C}.$$

But, in general, the definition of ρ_f only implies $\forall\, \varepsilon > 0$, $\exists\, A, B > 0$ s.t.

$$|f(z)| \le A e^{B|z|^{\rho_f + \varepsilon}}, \quad \forall z \in \mathbb{C}.$$

Thm 2.1 If f is an entire function and has an order of growth $\leq \rho$, then

(i) $n(r) \leq C r^{\rho}$ for some $C>0$ & sufficiently large r.

(ii) If $z_1, z_2, \cdots$ are the zeros of f with $z_k \neq 0$, then $\forall s > \rho$ we have

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|^s} < \infty .$$

Pf: If $f(0)=0$, then

$$F(z) = \frac{f(z)}{z^{l}}, \quad \text{where } l = \text{order of zero at } 0,$$

is an entire function & $F(0) \neq 0$. Then the assumption $\Rightarrow$

$$|F(z)| = \frac{|f(z)|}{|z|^{l}} \quad \text{is bounded in } \{|z| \leq 1\} \text{ and}$$

$$|F(z)| \leq |f(z)| \leq A e^{B|z|^{\rho}} \quad \text{for } \{|z|>1\}$$

Hence F also has an order of growth $\leq \rho$, with the same zeros $z_1, z_2, \cdots, z_k \neq 0$, as f.

And
$$n_f(r) = n_F(r) - l.$$

Therefore, we only need to show Thm 2.1 for entire function f with $f(0) \neq 0$.

If $f(0) \neq 0$, then we can apply formula (2) in the previous section:

$$\int_0^R n(r)\frac{dr}{r} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta - \log|f(0)|$$

Take $R = 2r$, we have

$$\int_r^{2r} n(t)\frac{dt}{t} \le \int_0^{2r} n(t)\frac{dt}{t}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta - \log|f(0)|$$

Since n is non-decreasing, $n(t) \ge n(r)$ $\forall\ t \in (r, 2r)$.

$$\therefore \quad \int_r^{2r} n(t)\frac{dt}{t} \ge n(r)\int_r^{2r}\frac{dt}{t} = n(r)\log 2.$$

$$\Rightarrow \quad n(r)\log 2 \le \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta - \log|f(0)|$$

$$\le \log\left[Ae^{B(2r)^\rho}\right] - \log|f(0)|$$

$$= (2^\rho B)r^\rho + \log\frac{A}{|f(0)|}$$

$\le C r^\rho$ for r sufficiently large. (Ex!)
for some $C > 0$

This proves part (i).

(so one can see that in general, C and how large r needed depends on the function f and ρ.)

To prove part (ii), we note that there is only finitely many zeros of f inside $\{|z_k|<1\}$ and $\{2^j \le |z_k| < 2^{j+1}\}$.

Then
$$\sum_{1 \le |z_k| \le 2^{N+1}} \frac{1}{|z_k|^s} = \sum_{j=0}^{N} \left(\sum_{2^j \le |z_k| < 2^{j+1}} \frac{1}{|z_k|^s} \right)$$

$$\le \sum_{j=0}^{N} \frac{1}{2^{js}} \# \{z_k : 2^j \le |z_k| < 2^{j+1}\}$$

$$\le \sum_{j=0}^{N} \frac{1}{2^{js}} n_r(2^{j+1})$$

by part (i)
$$\le C \sum_{j=0}^{N} \frac{1}{2^{js}} 2^{(j+1)\rho}$$

$$= 2^\rho C \sum_{j=0}^{N} \left(\frac{1}{2^{s-\rho}}\right)^j$$

(since $s>\rho$)
$$< 2^\rho C \sum_{j=0}^{\infty} \left(\frac{1}{2^{s-\rho}}\right)^j$$

$$< \infty$$

Letting $N \to +\infty$ & using $\#\{|z_k|<1\}$ finite $\Rightarrow$

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|^s} < \infty$$ ※

(used absolutely convergence and hence the series can be rearranged)

Note: $s>\rho$ is important in the proof.

(Can't be improved to $s=\rho$.)

eg 1 Let $f(z) = \sin \pi z = \dfrac{e^{i\pi z} - e^{-i\pi z}}{2}$.

Then $|f(z)| \leq e^{\pi|z|}$, $\forall z \in \mathbb{C}$. (Ex!)

ie. f has an order of growth ≤ 1.

On the other hand, if $\exists \rho > 0$, $A, B > 0$ s.t.

$$|f(z)| \leq A e^{B|z|^\rho}, \quad \forall z \in \mathbb{C}.$$

Then $\left| \dfrac{e^{-\pi y} - e^{\pi y}}{2} \right| = |f(iy)| \leq A e^{B|y|^\rho}$

$$\Rightarrow \quad 1 - e^{-2\pi y} \leq 2A e^{(By^\rho - \pi y)} \quad \text{for } y > 0$$

If $\rho < 1$, we have $1 \leq 2A \lim_{y \to +\infty} e^{By^\rho - \pi y} = 0$

which is a contradiction.

$\therefore \quad \rho_f = \inf \rho = 1$.

Note that the zeros are $n \in \mathbb{Z}$, the Thm 2.1 $\Rightarrow$

$$\sum_{n \neq 0} \frac{1}{|n|^s} < \infty \quad \text{for } s > 1.$$

But $\sum_{n \neq 0} \frac{1}{|n|^s}$ diverges for $s \leq 1$

eg 2 $\quad f(z) = \cos z^{\frac{1}{2}} = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{(2n)!}$

Then $\quad \rho_f = \frac{1}{2} \quad$ (Ex!)

f has zeros at $z_n = \left[(n+\frac{1}{2})\pi\right]^2$, and

$$\sum_{n\in\mathbb{Z}} \frac{1}{|z_n|^s} = \sum_{n\in\mathbb{Z}} \frac{1}{\left[(n+\frac{1}{2})\pi\right]^{2s}} \begin{cases} \text{converges if } s > \frac{1}{2} \\ \text{diverge otherwise.} \end{cases}$$

§3 Infinite Products

3.1 Generalities

Def Given $\{a_n\}_{n=1}^{\infty}$ ($a_n \in \mathbb{C}$), we say that the <u>infinite product</u> (or just product)

$$\prod_{n=1}^{\infty} (1+a_n) \quad \underline{\text{converges}}$$

if $\lim_{N\to\infty} \prod_{n=1}^{N} (1+a_n)$ exists.

<u>Remarks</u>: $\prod_{n=1}^{N} (1+a_n)$ is called the N-term partial product

<u>Prop 3.1</u>: $\sum |a_n| < \infty \Rightarrow \prod_{n=1}^{\infty} (1+a_n)$ converges

In this case, $\prod_{n=1}^{\infty} (1+a_n) = 0 \Leftrightarrow \exists\, n_0$ s.t. $1+a_{n_0} = 0$

<u>Pf</u>: $\sum |a_n| < \infty \Rightarrow |a_n| < \frac{1}{2}$ for sufficiently large n

$\Rightarrow$ for suff. large n, $\log(1+a_n) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{a_n^k}{k}$ is well-defined and

satisfies $\quad 1+a_n = e^{\log(1+a_n)}$ (actually holds $\forall\ |z|<1$)

Hence $\prod_{n=1}^{N} (1+a_n) = \prod_{n=1}^{N} e^{\log(1+a_n)} = e^{\sum_{n=1}^{N} \log(1+a_n)}$.

By the definition of $\log(1+a_n)$, we have for sufficiently large n,

$$|\log(1+a_n)| \leq 2|a_n| \quad \text{for } |a_n| < \frac{1}{2}$$

i.e. $$\sum_{n=1}^{N} |\log(1+a_n)| \leq 2\sum_{n=1}^{N} |a_n|$$

$$\sum |a_n| < \infty \Rightarrow \sum_{n=1}^{N} \log(1+a_n) \text{ converges absolutely}$$

$$\therefore \lim_{N\to+\infty} \prod_{n=1}^{N} (1+a_n) = e^{\sum_{n=1}^{\infty} \log(1+a_n)} \text{ exists.}$$

In this case, if $\exists n_0$ s.t. $1+a_{n_0} = 0$, then

$\lim_{N\to+\infty} \prod_{n>n_0}^{N} (1+a_n)$ exists and

$$\lim_{N\to\infty} \prod_{n=1}^{N} (1+a_n) = \prod_{n=1}^{n_0} (1+a_n) \cdot \lim_{N\to\infty} \prod_{n>n_0}^{N} (1+a_n) = 0$$

Since $\sum |a_n| < 0$, if $1+a_n \neq 0$, $\forall n$.

Then $$\lim_{N\to\infty} \prod_{n=1}^{N} (1+a_n) = e^{\sum_{n=1}^{\infty} \log(1+a_n)} \neq 0 . \quad \#$$

Prop 3.2 Suppose $\{F_n(z)\}$ is a seq. of holo. functions on Ω (open).
If $\exists\, C_n > 0$ such that

$$\begin{cases} \sum C_n < \infty & \text{and} \\ |F_n(z) - 1| \le C_n, & \forall z \in \Omega, \end{cases}$$

then
(i) $\prod_{n=1}^{\infty} F_n(z)$ converges unifomly in Ω to a holo. function $F(z)$.

(ii) If $F_n(z) \neq 0$, $\forall z \in \Omega$, $\forall n$, then

$$\frac{F'(z)}{F(z)} = \sum_{n=1}^{\infty} \frac{F_n'(z)}{F_n(z)}.$$

Pf: Write $F_n(z) = 1 + a_n(z)$.
Then by assumption $|a_n(z)| \le C_n$

and hence $\sum a_n(z)$ unifomly absolute converges on Ω.

By the same argument, as $N \to +\infty$,

$$G_N(z) = \prod_{n=1}^{N} F_n(z) \longrightarrow F(z) = e^{\sum_{n=1}^{\infty} \log(1+a_n(z))} \quad (\text{unifomly})$$

which has to be holomorphic on Ω. This proves (i).

For (ii), (Thm 5.3 of Ch 2)
$G_N \to F$ unifomly $\Rightarrow$

$G_N' \to F'$ unifomly on any cpt subset $K \subset \Omega$

By Prop 3.1, the limit $F(z) \neq 0$, $\forall z \in \Omega$.

for N large

Hence $\forall$ cpt. subset $K \subset \Omega$, $\exists \delta > 0$ s.t. $|G_N(z)| \geq \delta$.

$$\therefore \quad \sum_{n=1}^{N} \frac{F_n'(z)}{F_n(z)} = \frac{G_N'(z)}{G_N(z)} \to \frac{F'(z)}{F(z)} \text{ uniformly on } K.$$

Since $K \subset \Omega$ is arbitrary, we have $\displaystyle\frac{F'(z)}{F(z)} = \sum_{n=1}^{\infty} \frac{F_n'(z)}{F_n(z)}$.

\#

3.2 <u>Example: the product formula for the sine function</u>

$$\boxed{\frac{\sin \pi z}{\pi} = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \qquad \text{———} (3)}$$

We'll prove it by showing that

$$\boxed{\pi \cot \pi z = \lim_{N \to +\infty} \sum_{|n| \leq N} \frac{1}{z+n} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \qquad \text{———} (4)}$$

<u>Remarks</u>: (i) Formula (4) holds for $z \in \mathbb{C} \setminus \mathbb{Z}$ only

(ii) $\displaystyle\lim_{N \to +\infty} \sum_{|n| \leq N} \frac{1}{z+n}$ is the principal value of $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{z+n}$,

other arrangement may not converges.

Pf of (3) by (4).

Write $\quad G(z) = \dfrac{\sin \pi z}{\pi}$

$$P(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

$P(z)$ is well-defined since $\left|\dfrac{-z^2}{n^2}\right| = \dfrac{|z|^2}{n^2} \leq \dfrac{R^2}{n^2}$, $\forall z \in \{|z|<R\}$

Prop 3.2 $\Rightarrow$ $\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$ and hence $P(z)$ is well-defined on $\{|z|<R\}$. Since $R>0$ is arbitary, $P(z)$ is entire.

Again by Prop 3.2, for $z \in \mathbb{C} \setminus \mathbb{Z}$,

$$\frac{P'(z)}{P(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2} = \pi \cot \pi z \qquad \text{by formula (4)}.$$

Hence for $z \in \mathbb{C} \setminus \mathbb{Z}$

$$\left(\frac{P(z)}{G(z)}\right)' = \frac{P(z)}{G(z)}\left[\frac{P'(z)}{P(z)} - \frac{G'(z)}{G(z)}\right]$$

$$= \frac{P(z)}{G(z)}\left[\pi \cot \pi z - \frac{\cos \pi z}{\left(\frac{\sin \pi z}{\pi}\right)}\right] = 0$$

Since $\mathbb{C}\setminus\mathbb{Z}$ is connected, $\quad P(z) = c\,G(z)$ for some constant c.
(and clearly extends to whole $\mathbb{C}$)

Letting $z \to 0$ in $\dfrac{P(z)}{z} = c\,\dfrac{G(z)}{z}$ (near, but $\neq 0$),

i.e. $\displaystyle\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) = c\,\frac{\sin \pi z}{\pi z}$, we have $\quad c = 1$. #

Pf of formula (4)

Let $F(z) = \pi \cot \pi z$.

Then (i) $F(z+1) = F(z)$, $z \in \mathbb{C} \setminus \mathbb{Z}$

(ii) $F(z) = \frac{1}{z} + F_0(z)$, where F_0 analytic near 0.

(iii) $z = n \in \mathbb{Z}$ are simple pole of $F(z)$, &

$F(z)$ has no other singularities.

Note that

$$G(z) = \lim_{N\to+\infty} \sum_{|n|\leq N} \frac{1}{z+n} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}$$

clearly satisfies (ii).

And
$$G(z+1) = \lim_{N\to+\infty} \sum_{|n|\leq N} \frac{1}{z+1+n}$$

$$= \lim_{N\to+\infty} \left[\frac{1}{z+1-N} + \frac{1}{z-N+2} + \cdots + \frac{1}{z+N} + \frac{1}{z+1+N} \right]$$

$$= \lim_{N\to+\infty} \left[\left(\sum_{|n|\leq N} \frac{1}{z+n} \right) - \frac{1}{z-N} + \frac{1}{z+1+N} \right]$$

$$= G(z) \qquad \text{as } \lim_{N\to+\infty} \frac{1}{z+1+N} = 0 = \lim_{N\to+\infty} \frac{1}{z-N}.$$

This proves G also satisfies (i).

Then (i) and (ii) together implies (iii).

Now consider $\Delta(z) = F(z) - G(z)$.

Then by (i), $\Delta(z+1) = \Delta(z)$ (periodic)

By (ii) $\Delta(z) = \frac{1}{z} + F_0(z) - \frac{1}{z} - G_0(z)$ near $z=0$

$\left(\text{where } G_0(z) = \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2} \right)$

$= F_0(z) - G_0(z)$ analytic near $z=0$

$\therefore$ $z=0$ is a removable singularity of $\Delta(z)$.

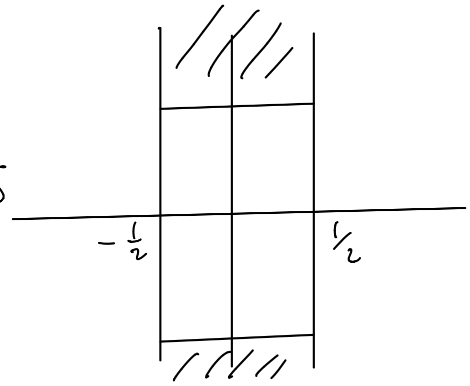
Together with (i) and (iii), all $z=n$ are removable singularities and hence $\Delta(z)$ is entire.

If $z = x+iy$ with $|x| \le \frac{1}{2}$ and $|y|>1$,

then
$$\cot \pi z = i \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} = i \frac{e^{-\pi y + i\pi x} + e^{\pi y - i\pi x}}{e^{-\pi y + i\pi x} - e^{\pi y - i\pi x}}$$
$$= i \frac{e^{-2\pi y} + e^{-2i\pi x}}{e^{-2\pi y} - e^{-2i\pi x}}$$

$$\therefore \quad |\cot \pi z|^2 = \frac{(e^{-2\pi y} + \cos 2\pi x)^2 + (\sin 2\pi x)^2}{(e^{-2\pi y} - \cos 2\pi x)^2 + (\sin 2\pi x)^2}$$
$$\le \frac{(e^{-2\pi y}+1)^2}{(e^{-2\pi y}-1)^2} \qquad (\text{Ex!})$$

$$\Rightarrow \quad |\cot \pi z| \le \frac{1+e^{-2\pi}}{1-e^{-2\pi}} \qquad \text{for all } |y|>1 \ \& \ |x| \le \tfrac{1}{2}$$



$\therefore$ $|F(z)| \le C$ on $\{|\mathrm{Re}\, z| \le \frac{1}{2}, |\mathrm{Im}\, z| > 1\}$ for some C.

Now for $G(z)$,

$$G(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}$$

$\Rightarrow$ On $\{z = x+iy : |x| \le \frac{1}{2} \ \& \ |y| > 1\}$,

$$|G(z)| \le \frac{1}{|z|} + \sum_{n=1}^{\infty} \left| \frac{2z}{z^2-n^2} \right|$$

$$\le 1 + C \sum_{n=1}^{\infty} \frac{|y|}{y^2+n^2} \qquad (\text{Ex!})$$

By Riemann sum

$$\sum_{n=1}^{\infty} \frac{|y|}{y^2+n^2} \le \int_0^{\infty} \frac{|y|}{y^2+x^2}\,dx$$

$$= \int_0^{\infty} \frac{|y|}{y^2+y^2t^2}\,|y|\,dt$$

$$= \int_0^{\infty} \frac{dt}{1+t^2} = B.$$

$\Rightarrow$ $|G(z)|$ is also bounded on $\{z = x+iy : |x| \le \frac{1}{2} \ \& \ |y| > 1\}$.

$\therefore$ $|\Delta(z)|$ is bounded on $\{z = x+iy : |x| \le \frac{1}{2} \ \& \ |y| > 1\}$.

Since $\Delta(z)$ is entire, it is bounded on

$$\{z = x+iy : |x| \le \tfrac{1}{2} \ \& \ |y| \le 1\}$$

Together we have $|\Delta(z)|$ is bounded on $\{z = x+iy : |x| \le \frac{1}{2}\}$.

Then by periodicity $\Delta(z+1) = \Delta(z)$, we conclude that

$\Delta(z)$ is bounded on $\mathbb{C}$.

Hence Liouville's Thm $\Rightarrow$ $\Delta(z) = \text{constant} = c$

Finally
$$c = \Delta(-z) = F(-z) - G(-z)$$
$$= \pi\cot(-\pi z) - \left[\frac{1}{-z} + \sum_{n=1}^{\infty} \frac{2(-z)}{(-z)^2 - n^2}\right]$$
$$= -\left[\pi\cot\pi z - \left(\frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}\right)\right]$$
$$= -\Delta(z) = -c$$

$\therefore\ c = 0$

$$\Rightarrow \quad \pi\cot\pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \qquad \#$$