# MATH4060 Assignment 3

Ki Fung, Chan

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1. Show that

   (a)
   $$\int_1^\infty e^{-t} t^{s-1} dt$$
   defines an entire function.

   (b) $\forall \epsilon > 0, \exists C > 0$ such that
   $$|s| \log |s| \leq C|s|^{1+\epsilon}$$
   $\forall s \in \mathbb{C} \backslash \{0\}$.

   *Proof.* (a) The function
   $$F_N(s) = \int_1^N e^{-t} t^{s-1} dt$$
   is entire for any $N > 1$. It suffices to show that $F_N$ converges uniformly on compact subsets. But for $|s| < R$, we have
   $$|F_N(s)| \leq \int_1^N e^{-t} t^R \leq C \int_1^\infty e^{-t/2} = \frac{C}{2}.$$

   (b) It is the same as to show that
   $$r \leq C e^{\epsilon r}$$
   for all $r > 0$. The right hand side is greater than $C(\epsilon r)$, so just take $C = \frac{1}{\epsilon}$.

   □

2. (a) Prove that
   $$\frac{d^2 \log \Gamma(s)}{ds^2} = \sum_{n=0}^\infty \frac{1}{(n+s)^2}$$
   for positive $s$. Show that if the left-hand side is interpreted as $(\Gamma'/\Gamma)'$, then the above formula holds for $s \neq 0, -1, -2, \ldots$.

   (b) Using part a), show that
   $$\Gamma(s)\Gamma(s + \frac{1}{2}) = \sqrt{\pi} 2^{1-2s} \Gamma(2s)$$

*Proof.* (a) We will use the formula:

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} (1 + \frac{z}{n}) e^{\frac{z}{n}}.$$

Taking the second derivative of $\log \Gamma(z)$, we have

$$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}.$$

(b) Now, we compute

$$\begin{aligned}
\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) + \frac{d}{dz}\left(\frac{\Gamma'(z+\frac{1}{2})}{\Gamma(z)+\frac{1}{2}}\right) &= \sum_{n=0}^{\infty} \frac{1}{(z+n)^2} + \sum_{n=0}^{\infty} \frac{1}{(z+n+\frac{1}{2})^2} \\
&= 4\left[\sum_{n=0}^{\infty} \frac{1}{(2z+2n)^2} + \sum_{n=0}^{\infty} \frac{1}{(2z+2n+1)^2}\right] \\
&= \sum_{n=0}^{\infty} \frac{4}{(2z+n)^2} \\
&= 4\frac{d}{dw}\left(\frac{\Gamma'(w)}{\Gamma(w)}\right)\Bigg|_{w=2z} \\
&= 2\frac{d}{dz}\left(\frac{\Gamma'(2z)}{\Gamma(2z)}\right).
\end{aligned}$$

Integration back, we have

$$\Gamma(z)\Gamma(z+\frac{1}{2}) = e^{az+b}\Gamma(2z),$$

for some constant $a, b$. Substituing $z = \frac{1}{2}$, and making use $\Gamma(\frac{1}{2}) = \sqrt{\pi}, \Gamma(1) = 1, \Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}, \Gamma(2) = 1$. We have

$$\begin{aligned}
\sqrt{\pi} &= e^{\frac{1}{2}a+b} \\
\frac{1}{2}\sqrt{\pi} &= e^{a+b}.
\end{aligned}$$

So we obtain

$$\begin{aligned}
e^a &= \frac{1}{4} \\
e^b &= 2\sqrt{\pi}
\end{aligned}$$

whence the result.

□

3. Let $f(z) = e^{az}e^{-e^z}$, $a > 0$. Observe that in the strip $\{x + iy : |y| < \pi/2\}$ the function $f(x + iy)$ is exponentially decreasing as $|x|$ tends to infinity. Prove that

$$\hat{f}(\xi) = \Gamma(a - 2\pi i\xi).$$

*Proof.* Using the substitution $t = e^x$,

$$\begin{aligned}\hat{f}(\xi) &= \int_{-\infty}^{\infty} e^{ax} e^{-e^x} e^{-2\pi i x \xi} dx \\ &= \int_{-\infty}^{\infty} e^{(a-2\pi i\xi)x} e^{-e^x} e^{-2\pi i x\xi} dx \\ &= \int_{0}^{\infty} t^{(a-2\pi i\xi)-1} e^{-t} dt \\ &= \Gamma(a - 2\pi i\xi).\end{aligned}$$

□

4. (a) Show that $1/|\Gamma(s)|$ is not $O(e^{c|s|})$ for any $c > 0$. [Hint: If $s = -k - 1/2$, where $k$ is a positive integer, then $|1/\Gamma(s)| \geq k!/\pi$.]

   (b) Show that there is no entire function $F(s)$ with $F(s) = O(e^{c|s|})$ that has simple zeros at $s = 0, -1, -2, \ldots, -n, \ldots$, and that vanishes nowhere else.

*Proof.* (a) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, and hence

$$\Gamma(-k - \frac{1}{2}) = \frac{\sqrt{\pi}}{(-\frac{1}{2})(-1-\frac{1}{2})\cdots(-k-\frac{1}{2})}$$

So,

$$\left|\frac{1}{\Gamma(-k-\frac{1}{2})}\right| = \frac{k!}{\sqrt{\pi}}.$$

The result follows from the well-known fact that

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0$$

for any $a > 0$. This fact can be proved by calculating the ratios: $\frac{a^{n+1}/(n+1)!}{a^n/n!} = \frac{a}{n+1} < \frac{1}{2}$ for all large enough $n$.

(a) For if such an $F$ exists, then by the Hadamard factorization

$$F(s) = e^{Az+B} z \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$$

In other words, we have

$$\frac{1}{\Gamma(z)} = F(z) e^{A'z+B},$$

this contradicts to (a), because the right hand side has growth order 1.

□

5. Prove that for $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx.$$

[Hint: Write $1/(e^x - 1) = \sum_{n=1}^\infty e^{-nx}$.]

*Proof.*

$$\begin{aligned} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx &= \sum_{n=1}^\infty \int_0^\infty x^{s-1} e^{-nx} dx \\ &= \sum_{n=1}^\infty \frac{1}{n^s} \int_0^\infty y^{s-1} e^{-y} dy \\ &= \sum_{n=1}^\infty \frac{1}{n^s} \Gamma(s). \end{aligned}$$

Whence the result. □