1. **Curves in $\mathbb{R}^3$**

Definition 1.1. A (*parametrized smooth*) *curve* $\alpha(t)$ is a smooth map

$$\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$$

from an interval I in $\mathbb{R}$ into $\mathbb{R}^3$ so that α is smooth. α is said to be *regular* if $\alpha' \neq 0$.

Let $\alpha : (a, b) \to \mathbb{R}^3$ is a curve. Let $f : (c, d) \to (a, b)$ with $t = f(\sigma)$ such that $f' > 0$, then $\alpha(f(\sigma)) : (c, d) \to \mathbb{R}^3$ is said to be a *reparametrization* of α.

Let α be a regular curve defined on $[a, b]$ and let $t_0 \in [a, b]$, the *arc-length* is defined as:

$$s(t) = \int_{t_0}^{t} |\alpha'(u)| du.$$

If $s(a) = -L_1, s(b) = L_2$, then $\alpha(s) = \alpha(s(t))$ is a reparametrization of α and $\alpha(s)$ is said to be parametrized by arc-length.

Fact: $\alpha = \alpha(t)$ is parametrized by arc-length, that is t 'represents' arc-length from a fixed point iff $|\alpha'| = 1$.

2. **The Frenet formula**

Let $\alpha(s)$ be the regular curve parametrized by arc length.

Let $\vec{T} = \alpha'$. Then

$$\begin{aligned} k(s) :=& |T'|(s) \quad \text{(curvature)}; \\ N(s) :=& \frac{1}{k(s)} T'(s) \quad \text{(normal, if } k > 0); \\ B(s) :=& T(s) \times N(s) \quad \text{(binormal, if } k > 0). \end{aligned}$$

Fact: $B' = -\tau N$, τ is called the torsion of α.

Theorem 2.1. *(Frenet formula) Let α be a regular curve with curvature $k > 0$. Then*

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

We summarize some properties on curves:

Theorem 2.2. *Let α be a regular curves in $\mathbb{R}^3$ parametrized by arc length.*

(i) *Suppose the curvature $k \equiv 0$ if and only if α is a straight line.*
(ii) *Suppose the curvature $k > 0$ and the torsion $\tau \equiv 0$ if and only if α is a plane curve.*

(iii) *Suppose the curvature* $k = k_0 > 0$ *is a constant and* $\tau \equiv 0$, *then* α *is a circular arc with radius* $1/k_0$.
(iv) *Suppose the curvature* $k > 0$ *and the torsion* $\tau \neq 0$ *everywhere.* α *lies on s sphere if and only if* $\rho^2 + (\rho')^2\lambda^2$ =*constant, where* $\rho = 1/k$ *and* $\lambda = 1/\tau$.
(v) *Suppose the curvature* $k = k_0 > 0$ *is a constant and* $\tau = \tau_0$ *is a constant. Then* α *is a circular helix.*
(vi) *Suppose* α *is defined on* $[a, b]$. *Let* $\mathbf{p} = \alpha(a)$ *and* $\mathbf{q} = \alpha(b)$. *Then the length* l *of* α *satisfies* $l \leq |\mathbf{p} - \mathbf{q}|$. *Moreover, equality holds if and only if* α *is the straight line from* $\mathbf{p}$ *to* $\mathbf{q}$.

3. Review: Existence and uniqueness thoerems in ODE

Ref: *Ordinary differential equations, Birkoff and Rota*

We only consider the special case of linear ODE. Let $A(t) = (a_{ij}(t))_{n\times n}$ be a smooth family $n \times n$ matrix, $t \in [a, b]$. Consider the following initial valued problem (IVP): Given A and a constant $\mathbf{x}_0 \in \mathbb{R}^n$, to find $\mathbf{x} : [a, b] \to \mathbb{R}^n$ satisfying:

$$\begin{cases} \mathbf{x}'(t) & = A(t)\mathbf{x}(t), \quad t \in [a, b]; \\ \mathbf{x}(a) & = \mathbf{x}_0. \end{cases}$$

Theorem 3.1. *Given any* $\mathbf{x}_0 \in \mathbb{R}^n$, *the exists a unique solution of the above IVP.*

Proof. (Sketch) For simplicity let us assume $a = 0$.

Existence: Define inductively, with $\mathbf{x}_0(t) = \mathbf{x}_0$ for all t, and

$$\mathbf{x}_{k+1}(t) = \mathbf{x}_0 + \int_0^t A(\tau)\mathbf{x}_k(\tau)d\tau.$$

for $k \geq 0$. Let $M = \sup_{t\in[a,b]} ||A||(t)$ and $||A(t)||^2 = \mathrm{tr}(\mathrm{AA^T(t)})$. For $k \geq 1$, we have

$$|\mathbf{x}_{k+1}(t) - \mathbf{x}_k(t)| \leq M \int_0^t |\mathbf{x}_k(\tau) - \mathbf{x}_{k-1}(\tau)|d\tau.$$

Inductively, we have (why?)

$$\begin{aligned} |\mathbf{x}_{k+1}(t) - \mathbf{x}_k(t)| \leq & M^k \int_0^t \int_0^{\tau_{k-1}} \cdots \int_0^{\tau_2} \int_0^{\tau_1} |\mathbf{x}_1(\tau_1) - \mathbf{x}_0(\tau_1)|d\tau_1 d\tau_2 \ldots d\tau_{k-1} d\tau_k \\ \leq & \frac{M^k b^k S}{k!} \end{aligned}$$

where integration is over the domain $t \geq \tau_k \geq \cdots \geq \tau_1$ and $S = \sup_{t\in[0,b]} |\mathbf{x}_1(t) - \mathbf{x}_0(t)|$.

Hence $\sum_{k=1}^{\infty}|\mathbf{x}_{k+1}(t)-\mathbf{x}_k(t)|\le C$ for some constant C for all $t\in[0,b]$. This implies that $\mathbf{x}_k\to\mathbf{x}_\infty$ uniformly on $[0,b]$ which satisfies:

$$\mathbf{x}_\infty(t)=\mathbf{x}_0+\int_0^t A(\tau)\mathbf{x}_\infty(\tau)d\tau,$$

(why?) Now $\mathbf{x}_\infty$ is the solution of the above IVP.

Uniquess: Sufficient to prove that if $\mathbf{x}_0=\mathbf{0}$, then any solution must be trivial. So let $\mathbf{x}$ be such a solution, then

$$\frac{d}{dt}||\mathbf{x}||^2=2\langle A\mathbf{x},\mathbf{x}\rangle\le 2M||\mathbf{x}||^2.$$

Hence

$$\frac{d}{dt}\left(\exp(-2Mt)||\mathbf{x}||^2\right)\le 0.$$

This will imply that $||\mathbf{x}||^2\equiv 0$. (Why?) □

4. Fundamental theorem of the local theorey of curves

Theorem 4.1. *Let $\kappa(s)>0$ and $\tau(s)$ be smooth function on (a,b). There exists a regular curve $\alpha:(a,b)\to\mathbb{R}^3$ with $|\alpha'|=1$, such that the curvature and torsion of α are k, τ respectively.*

Moreover, α is unique in the sense that if β is another curve satisfying the above conditions, then $\beta(s)=\alpha(s)P+\vec{c}$ for some constant orthogonal matrix P and some constant vector $\vec{c}$. **Here α,β are considered as row vectors.**

Proof. (**Existence**): Let

$$A(s)=\begin{pmatrix}0&\kappa(s)&0\\-\kappa(s)&0&\tau(s)\\0&-\tau(s)&0\end{pmatrix}.$$

Let $X(s)$ be the 3×3 matrix and fix s_0 which is the solution of:

$$\begin{cases}X'&=AX\quad\text{in }(a,b);\\X(s_0)&=I.\end{cases}$$

The solution exists by a theorem in ODE. We claim that X is orthogonal with determinant 1. In fact

$$(X^tX)'=(X^t)'X+X^tX'=(AX)^tX+X^tAX=X^tA^tX+X^tAX=0$$

because $A^t=-A$. Hence $X^tX=I$ because $X^t(s_0)X(s_0)=I$. (**Using $(XX^t)'$ may be more involved.**) Hence $X(s)$ is orthogonal. Since

$\det X(s) = 1$ or -1 and initially, $\det X(s_0) = 1$, we have $\det X(s) = 1$. Write

$$X = \begin{pmatrix} \widetilde{T} \\ \widetilde{N} \\ \widetilde{B} \end{pmatrix}.$$

Define $\alpha(s) = \int_{s_0}^{s} \widetilde{T}(\sigma)d\sigma$. Let T, N, B be the tangent, principal normal and binormal of α, and let $\kappa_\alpha, \tau_\alpha$ be the curvature and torsion of α. Then $\alpha' = \widetilde{T}$ which has length 1. So $T = \widetilde{T}$. Moreover,

$$\kappa_\alpha N = T' = \widetilde{T}' = \kappa \widetilde{N}.$$

we have $\kappa_\alpha = \kappa$ and $N = \widetilde{N}$. Since $\widetilde{T}, \widetilde{N}, \widetilde{B}$ are positively oriented, we conclude that

$$B = T \times N = \widetilde{T} \times \widetilde{N} = \widetilde{B},$$

and

$$-\tau_\alpha N = B' = \widetilde{B}' = -\tau \widetilde{N} = -\tau N.$$

Hence $\tau_\alpha = \tau$.

Uniqness: Let α, β as in the theorem. Let $T_\alpha, N_\alpha, B_\alpha$ be the unit tangent, principal normal, binormal of α; and let $T_\beta, N_\beta, B_\beta$ be the unit tangent, principal normal, binormal of β. Fix $s_0 \in (a, b)$. Let P be an orthogonal matrix with determinant 1 such that

$$\begin{pmatrix} T_\beta(s_0) \\ N_\beta(s_0) \\ B_\beta(s_0) \end{pmatrix} = \begin{pmatrix} T_\alpha(s_0) \\ N_\alpha(s_0) \\ B_\alpha(s_0) \end{pmatrix} P.$$

Here $T_\alpha, \dots,$ etc are considered as row vectors. Let $\gamma(s) = \alpha(s)P$. Let T_γ, N_γ, B_γ be unit tangent, principal normal, binormal of γ. Then

$$T_\gamma = \gamma' = \alpha' P = T_\alpha P,$$

$$\kappa N_\gamma = T_\gamma' = T_\alpha' P = \kappa N P.$$

and so $T_\gamma = T_\alpha P, N_\gamma = N_\alpha P$. Hence $B_\gamma = B_\alpha P$. We have

$$\begin{pmatrix} T_\gamma \\ N_\gamma \\ B_\gamma \end{pmatrix}' = \begin{pmatrix} T_\alpha \\ N_\alpha \\ B_\alpha \end{pmatrix}' P = A \begin{pmatrix} T_\alpha \\ N_\alpha \\ B_\alpha \end{pmatrix} P = A \begin{pmatrix} T_\gamma \\ N_\gamma \\ B_\gamma \end{pmatrix}$$

where A is as above. Since

$$\begin{pmatrix} T_\gamma(s_0) \\ N_\gamma(s_0) \\ B_\gamma(s_0) \end{pmatrix} = \begin{pmatrix} T_\alpha(s_0) \\ N_\alpha(s_0) \\ B_\alpha(s_0) \end{pmatrix} P = \begin{pmatrix} T_\beta(s_0) \\ N_\beta(s_0) \\ B_\beta(s_0) \end{pmatrix}.$$

we have $T_\gamma = T_\beta$, by uniqueness theorem of ODE. So $\gamma(s) + \vec{c} = \beta(s)$ for some constant vector $\vec{c}$. That is: $\beta(s) = \alpha(s)P + \vec{c}$.

□

5. Geometric meaning of curvature

Proposition 5.1. *Let $\alpha(s)$ be a plane curve parametrized by arc length defined on (a,b). Let $s_0 \in (a,b)$. Suppose $\kappa(s_0) > 0$. Then the following are true:*

(i) *For any $s_1 < s_2 < s_3$ sufficiently close to s_0, $\alpha(s_1), \alpha(s_2), \alpha(s_3)$ are not collinear.*

(ii) *For $s_1 < s_2 < s_3$ sufficiently close to s_0 so that $\alpha(s_1), \alpha(s_2), \alpha(s_3)$ are not collinear, let $c(s_1, s_2, s_3)$ be the center of the unique circle $C(s_1, s_2, s_3)$ passing through $\alpha(s_1), \alpha(s_2), \alpha(s_3)$. As $s_1, s_2, s_3 \to s_0$, $C(s_1, s_2, s_3)$ will converge to a circle passing through $\alpha(s_0)$ tangent to α at $\alpha(s_0)$ with radius $1/\kappa(s_0)$*

Proof. (i) Suppose $\alpha(s_1), \alpha(s_2), \alpha(s_3)$ lie on a straight line. Then

$$\langle \alpha(s_i) - \vec{v}, \vec{n} \rangle = 0$$

for some constant vectors $\vec{v}, \vec{n}$ with $|\vec{n}| = 1$, for $i = 1, 2, 3$. Let $f(s) = \langle \alpha(s) - \vec{v}, \vec{n} \rangle$. Then $f(s_i) = 0$ for $i = 1, 2, 3$. Hence $f'(\xi_1) = f'(\xi_2) = 0$ for some $s_1 < \xi_1 < s_2 < \xi_2 < s_3$ and $f''(\eta) = 0$ for some $\xi_1 < \eta < \xi_2$. That is:

$$\begin{cases} \langle \alpha'(\xi_1), \vec{n} \rangle = & \langle \alpha'(\xi_2), \vec{n} \rangle = 0; \\ \langle \alpha''(\eta), \vec{n} \rangle = & 0. \end{cases}$$

As $s_1, s_2, s_3 \to s_0$, $\vec{n} \to N(s_0)$ and $\alpha''(\eta) = \kappa(s_0) N(s_0)$. This implies $\kappa(s_0) = 0$. Contradiction.

(ii) Let $C(s_1, s_2, s_3)$ be given by

$$||\mathbf{x} - c|| = r.$$

where $c = c(s_1, s_2, s_3)$.

Let $h(s) = ||\alpha(s) - c||^2$. Then $h(s_i) = r^2$ for $i = 1, 2, 3$. Hence $h'(\xi_1) = h'(\xi_2) = 0$ for some $s_1 < \xi_1 < s_2 < \xi_2 < s_3$ and $h''(\eta) = 0$ for some $\xi_1 < \eta < \xi_2$. Hence

$$\begin{cases} \langle \alpha'(\xi_1), \alpha(\xi_1) - c \rangle & = \langle \alpha'(\xi_2), \alpha(\xi_2) - c \rangle = 0; \\ \langle \alpha''(\eta), \alpha(\eta) - c \rangle + 1 & = 0. \end{cases}$$

If $c \to c_\infty$ for some sequence $s_1 < s_2 < s_3 \to s_0$, then

$$\langle \alpha'(s_0), \alpha(s_0) - c_\infty \rangle = 0, \quad \langle \alpha''(s_0), \alpha(s_0) - c_\infty \rangle = -1$$

So $c_\infty - \alpha(s_0) = \frac{1}{\kappa(s_0)} N(s_0)$. From this the result follows.

□

The limiting circle is called the *osculating circle*.

6. Curvature and torsion in general parameter

Proposition 6.1. *Let $\alpha(t)$ be a regular curve with nonzero curvature. Then the curvature and torsion are given by:*

$$\begin{cases} \kappa = & \frac{|\alpha'\times\alpha''|}{|\alpha'|^3} \\ \tau = & \frac{<\alpha'\times\alpha'',\alpha'''>}{|\alpha'\times\alpha''|^2}. \end{cases}$$

Here $'$ always means differentiation with respect to t.

Proof. Let $\alpha(t)$ be a regular curve with nonzero curvature. Then

$$\alpha' = |\alpha'|T,$$

$$\alpha'' = \kappa|\alpha'|^2 N + |\alpha'|^{-1} < \alpha', \alpha'' > T. \tag{1}$$

Hence

$$< \alpha'', \alpha'' >= \kappa^2|\alpha'|^4 + |\alpha'|^{-2} < \alpha', \alpha'' >^2,$$

and

$$\begin{aligned} \kappa^2 &= \frac{< \alpha'', \alpha'' >< \alpha', \alpha' > - < \alpha', \alpha'' >^2}{|\alpha'|^6} \\ &= \frac{|\alpha' \times \alpha''|^2}{|\alpha'|^6}. \end{aligned}$$

To compute τ, note that

$$\alpha''' = \kappa(-kT + \tau B)|\alpha'|^3 + f(t)T + g(t)N$$

for some function f and g. (Why?). So

$$\tau = \frac{1}{\kappa}\frac{< \alpha''', B >}{|\alpha'|^3}.$$

Use (1)

$$\begin{aligned} B &= T \times N \\ &= \frac{T \times \alpha''}{k|\alpha'|^2} \\ &= \frac{\alpha' \times \alpha''}{k|\alpha'|^3} \end{aligned}$$

Use the formula for k, we have

$$\tau = \frac{< \alpha' \times \alpha'', \alpha''' >}{|\alpha' \times \alpha''|^2}.$$

□